

Problem 14

Find a formula involving integrals for a particular solution of the differential equation

$$y''' - y'' + y' - y = g(t).$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of $y_c(t)$ and $y_p(t)$, the complementary solution and the particular solution, respectively.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c''' - y_c'' + y_c' - y_c = 0 \tag{1}$$

Since each term on the left has constant coefficients, the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt} \quad \rightarrow \quad y_c''' = r^3e^{rt}$$

Substitute these expressions into the ODE.

$$r^3e^{rt} - r^2e^{rt} + re^{rt} - e^{rt} = 0$$

Divide both sides by e^{rt} .

$$r^3 - r^2 + r - 1 = 0$$

$$(r - 1)(r^2 + 1) = 0$$

$$r = \{1, -i, i\}$$

Three solutions to equation (1) are then $y_c = e^t$ and $y_c = e^{-it}$ and $y_c = e^{it}$. By the principle of superposition, the general solution for y_c is a linear combination of these three.

$$\begin{aligned} y_c(t) &= C_1e^t + C_2e^{-it} + C_3e^{it} \\ &= C_1e^t + C_2(\cos t - i \sin t) + C_3(\cos t + i \sin t) \\ &= C_1e^t + (C_2 + C_3) \cos t + (-iC_2 + iC_3) \sin t \\ &= C_1e^t + C_4 \cos t + C_5 \sin t \end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p''' - y_p'' + y_p' - y_p = g(t). \tag{2}$$

According to the method of variation of parameters, the particular solution can be obtained by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_1(t)e^t + C_4(t) \cos t + C_5(t) \sin t$$

Substitute this formula into equation (2).

$$\begin{aligned} &[C_1(t)e^t + C_4(t) \cos t + C_5(t) \sin t]''' - [C_1(t)e^t + C_4(t) \cos t + C_5(t) \sin t]'' \\ &+ [C_1(t)e^t + C_4(t) \cos t + C_5(t) \sin t]' - [C_1(t)e^t + C_4(t) \cos t + C_5(t) \sin t] = g(t) \end{aligned}$$

Evaluate the derivatives.

$$\begin{aligned}
 & [C_1'(t)e^t + C_1(t)e^t + C_4'(t)\cos t - C_4(t)\sin t + C_5'(t)\sin t + C_5(t)\cos t]'' \\
 & \quad - [C_1'(t)e^t + C_1(t)e^t + C_4'(t)\cos t - C_4(t)\sin t + C_5'(t)\sin t + C_5(t)\cos t]' \\
 & \quad + [C_1'(t)e^t + C_1(t)e^t + C_4'(t)\cos t - C_4(t)\sin t + C_5'(t)\sin t + C_5(t)\cos t] \\
 & \quad \quad - [C_1(t)e^t + C_4(t)\cos t + C_5(t)\sin t] = g(t)
 \end{aligned}$$

If we set $C_1'(t)e^t + C_4'(t)\cos t + C_5'(t)\sin t = 0$, then this equation simplifies to

$$\begin{aligned}
 & [C_1(t)e^t - C_4(t)\sin t + C_5(t)\cos t]'' - [C_1(t)e^t - C_4(t)\sin t + C_5(t)\cos t]' \\
 & \quad + [C_1(t)e^t - C_4(t)\sin t + C_5(t)\cos t] - [C_1(t)e^t + C_4(t)\cos t + C_5(t)\sin t] = g(t)
 \end{aligned}$$

$$\begin{aligned}
 & [C_1'(t)e^t + C_1(t)e^t - C_4'(t)\sin t - C_4(t)\cos t + C_5'(t)\cos t - C_5(t)\sin t]' \\
 & \quad - [C_1'(t)e^t + C_1(t)e^t - C_4'(t)\sin t - C_4(t)\cos t + C_5'(t)\cos t - C_5(t)\sin t] \\
 & \quad + [C_1(t)e^t - C_4(t)\sin t + C_5(t)\cos t] - [C_1(t)e^t + C_4(t)\cos t + C_5(t)\sin t] = g(t).
 \end{aligned}$$

If we set $C_1'(t)e^t - C_4'(t)\sin t + C_5'(t)\cos t = 0$, then this equation simplifies to

$$\begin{aligned}
 & [C_1(t)e^t - C_4(t)\cos t - C_5(t)\sin t]' - [C_1(t)e^t - C_4(t)\cos t - C_5(t)\sin t] \\
 & \quad + [C_1(t)e^t - C_4(t)\sin t + C_5(t)\cos t] - [C_1(t)e^t + C_4(t)\cos t + C_5(t)\sin t] = g(t)
 \end{aligned}$$

$$\begin{aligned}
 & [C_1'(t)e^t + \cancel{C_1(t)e^t} - C_4'(t)\cos t + \cancel{C_4(t)\sin t} - C_5'(t)\sin t - C_5(t)\cos t] \\
 & \quad - [\cancel{C_1(t)e^t} - \cancel{C_4(t)\cos t} - C_5(t)\sin t] \\
 & \quad + [\cancel{C_1(t)e^t} - \cancel{C_4(t)\sin t} + C_5(t)\cos t] - [\cancel{C_1(t)e^t} + \cancel{C_4(t)\cos t} + C_5(t)\sin t] = g(t) \\
 & \quad \quad C_1'(t)e^t - C_4'(t)\cos t - C_5'(t)\sin t = g(t).
 \end{aligned}$$

As a result of using the method of variation of parameters, the problem of finding a particular solution has reduced to solving the following system of ODEs.

$$C_1'(t)e^t + C_4'(t)\cos t + C_5'(t)\sin t = 0 \tag{3}$$

$$C_1'(t)e^t - C_4'(t)\sin t + C_5'(t)\cos t = 0 \tag{4}$$

$$C_1'(t)e^t - C_4'(t)\cos t - C_5'(t)\sin t = g(t) \tag{5}$$

Add the respective sides of equations (3) and (5) together.

$$2C_1'(t)e^t = g(t)$$

$$C_1'(t) = \frac{1}{2}e^{-t}g(t)$$

Integrate both sides with respect to t , setting the integration constant to zero.

$$C_1(t) = \int^t \frac{1}{2}e^{-s}g(s) ds$$

Subtract the respective sides of equation (4) from those of equation (3). Also, subtract the respective sides of equation (5) from those of equation (3).

$$\begin{aligned}C_4'(t)(\cos t + \sin t) + C_5'(\sin t - \cos t) &= 0 \\2C_4'(t) \cos t + 2C_5'(t) \sin t &= -g(t)\end{aligned}$$

Solve this first equation for $C_4'(t)$

$$C_4'(t) = \frac{\cos t - \sin t}{\cos t + \sin t} C_5'(t) \quad (6)$$

and then plug it into the second equation.

$$2 \left[\frac{\cos t - \sin t}{\cos t + \sin t} C_5'(t) \right] \cos t + 2C_5'(t) \sin t = -g(t)$$

Divide both sides by 2 and factor $C_5'(t)$.

$$\begin{aligned}C_5'(t) \left(\frac{\cos t - \sin t}{\cos t + \sin t} \cos t + \sin t \right) &= -\frac{1}{2}g(t) \\C_5'(t) \left[\frac{\cos t - \sin t}{\cos t + \sin t} \cos t + \frac{\sin t(\cos t + \sin t)}{\cos t + \sin t} \right] &= -\frac{1}{2}g(t) \\C_5'(t) \left(\frac{\cos^2 t - \cancel{\sin t \cos t} + \cancel{\sin t \cos t} + \sin^2 t}{\cos t + \sin t} \right) &= -\frac{1}{2}g(t) \\C_5'(t) \left(\frac{1}{\cos t + \sin t} \right) &= -\frac{1}{2}g(t) \\C_5'(t) &= -\frac{1}{2}g(t)(\cos t + \sin t)\end{aligned}$$

Integrate both sides with respect to t , setting the integration constant to zero.

$$C_5(t) = \int^t -\frac{1}{2}g(s)(\cos s + \sin s) ds$$

Now substitute the previous equation into equation (6).

$$\begin{aligned}C_4'(t) &= \frac{\cos t - \sin t}{\cos t + \sin t} \left[-\frac{1}{2}g(t)(\cos t + \sin t) \right] \\&= -\frac{1}{2}g(t)(\cos t - \sin t)\end{aligned}$$

Integrate both sides with respect to t , setting the integration constant to zero.

$$C_4(t) = \int^t -\frac{1}{2}g(s)(\cos s - \sin s) ds$$

The particular solution is then

$$\begin{aligned}
 y_p(t) &= C_1(t)e^t + C_4(t) \cos t + C_5(t) \sin t \\
 &= \left[\int^t \frac{1}{2} e^{-s} g(s) ds \right] e^t + \left[\int^t -\frac{1}{2} g(s)(\cos s - \sin s) ds \right] \cos t + \left[\int^t -\frac{1}{2} g(s)(\cos s + \sin s) ds \right] \sin t \\
 &= \frac{1}{2} \int^t e^{t-s} g(s) ds - \frac{1}{2} \int^t g(s)(\cos t \cos s - \cos t \sin s) ds - \frac{1}{2} \int^t g(s)(\sin t \cos s + \sin t \sin s) ds \\
 &= \frac{1}{2} \int^t (e^{t-s} - \cos t \cos s + \cos t \sin s - \sin t \cos s - \sin t \sin s) g(s) ds \\
 &= \frac{1}{2} \int^t [e^{t-s} - (\cos t \cos s + \sin t \sin s) - (\sin t \cos s - \cos t \sin s)] g(s) ds \\
 &= \frac{1}{2} \int^t [e^{t-s} - \cos(t-s) - \sin(t-s)] g(s) ds,
 \end{aligned}$$

and the general solution is

$$\begin{aligned}
 y(t) &= y_c(t) + y_p(t) \\
 &= C_1 e^t + C_4 \cos t + C_5 \sin t + \frac{1}{2} \int^t [e^{t-s} - \cos(t-s) - \sin(t-s)] g(s) ds.
 \end{aligned}$$