Problem 12

In each of Problems 7 and 8, find the general solution of the given differential equation. Leave your answer in terms of one or more integrals.

\[ y''' - y' = \csc t; \quad y(\pi/2) = 2, \quad y'(\pi/2) = 1, \quad y''(\pi/2) = -1 \]

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of \( y_c(t) \) and \( y_p(t) \), the complementary solution and the particular solution, respectively.

\[ y(t) = y_c(t) + y_p(t) \]

The complementary solution satisfies the associated homogeneous equation.

\[ y'''_c - y'_c = 0 \]  \hspace{1cm} (1)

Since each term on the left has constant coefficients, the solution is of the form \( y_c = e^{rt} \).

\[ y_c = e^{rt} \quad \rightarrow \quad y'_c = re^{rt} \quad \rightarrow \quad y''_c = r^2 e^{rt} \quad \rightarrow \quad y'''_c = r^3 e^{rt} \]

Substitute these expressions into the ODE.

\[ r^3 e^{rt} - re^{rt} = 0 \]

Divide both sides by \( e^{rt} \).

\[ r^3 - r = 0 \]

\[ r(r^2 - 1) = 0 \]

\[ r = \{-1, 0, 1\} \]

Three solutions to equation (1) are then \( y_c = e^{-t} \) and \( y_c = e^0 = 1 \) and \( y_c = e^t \). By the principle of superposition, the general solution for \( y_c \) is a linear combination of these three.

\[ y_c(t) = C_1 e^{-t} + C_2 + C_3 e^t \]

On the other hand, the particular solution satisfies

\[ y'''_p - y'_p = \csc t. \]  \hspace{1cm} (2)

According to the method of variation of parameters, the particular solution can be obtained by allowing the parameters in \( y_c(t) \) to vary.

\[ y_p(t) = C_1(t)e^{-t} + C_2(t) + C_3(t)e^t \]

Substitute this formula into equation (2).

\[ [C_1(t)e^{-t} + C_2(t) + C_3(t)e^t]''' - [C_1(t)e^{-t} + C_2(t) + C_3(t)e^t]' = \csc t \]

Evaluate the derivatives.

\[ [C_1'(t)e^{-t} - C_1(t)e^{-t} + C_2'(t) + C_3'(t)e^t]' - [C_1'(t)e^{-t} - C_1(t)e^{-t} + C_2'(t) + C_3'(t)e^t]' = \csc t \]

www.stemjock.com
If we set \( C_1'(t)e^{-t} + C_2'(t) + C_3'(t)e^t = 0 \), then this equation simplifies to
\[
\left[ -C_1(t)e^{-t} + C_3(t)e^t \right]' - \left[ -C_1(t)e^{-t} + C_3(t)e^t \right] = \csc t
\]
\[
\left[ -C_1'(t)e^{-t} + C_1(t)e^{-t} + C_3'(t)e^t + C_3(t)e^t \right]' - \left[ -C_1(t)e^{-t} + C_3(t)e^t \right] = \csc t.
\]
If we set \( -C_1'(t)e^{-t} + C_3'(t)e^t = 0 \), then this equation simplifies to
\[
\left[ C_1(t)e^{-t} + C_3(t)e^t \right]' - \left[ -C_1(t)e^{-t} + C_3(t)e^t \right] = \csc t
\]
\[
\left[ C_1'(t)e^{-t} - C_1(t)e^{-t} + C_3(t)e^t + C_3(t)e^t \right]' - \left[ -C_1(t)e^{-t} + C_3(t)e^t \right] = \csc t
\]
\[
C_1'(t)e^{-t} + C_3'(t)e^t = \csc t.
\]
As a result of using the method of variation of parameters, the problem of finding a particular solution has reduced to solving the following system of ODEs.
\begin{align*}
C_1'(t)e^{-t} + C_2'(t) + C_3'(t)e^t &= 0 \\
-C_1'(t)e^{-t} + C_3'(t)e^t &= 0 \\
C_1'(t)e^{-t} + C_3'(t)e^t &= \csc t
\end{align*}
(3) \hspace{1cm} (4) \hspace{1cm} (5)
Start by solving equation (4) for \( C_1'(t) \).
\[
C_1'(t) = e^{2t}C_3'(t)
\]
and then plugging it in to equation (5).
\[
\left[ e^{2t}C_3'(t) \right]e^{-t} + C_3'(t)e^t = \csc t
\]
\[
2C_3'(t)e^t = \csc t
\]
\[
C_3'(t) = \frac{1}{2}e^{-t}\csc t
\]
Integrate both sides with respect to \( t \), setting the integration constant to zero.
\[
C_3(t) = \int^t \frac{1}{2}e^{-s}\csc s \, ds
\]
Substitute this back into equation (4) to get \( C_1(t) \).
\[
-C_1'(t)e^{-t} + C_3'(t)e^t = 0 \quad \rightarrow \quad -C_1'(t)e^{-t} + \left[ \frac{1}{2}e^{-t}\csc t \right]e^t = 0 \quad \rightarrow \quad -C_1'(t)e^{-t} + \frac{1}{2}\csc t = 0
\]
\[
C_1'(t) = \frac{1}{2}e^t\csc t
\]
Integrate both sides with respect to \( t \), setting the integration constant to zero.
\[
C_1(t) = \int^t \frac{1}{2}e^s\csc s \, ds
\]
Substitute this result along with \( C_3(t) \) into equation (3) to obtain \( C_2(t) \).
\[
C_1'(t)e^{-t} + C_2'(t) + C_3'(t)e^t = 0 \quad \rightarrow \quad \left[ \frac{1}{2}e^t\csc t \right]e^{-t} + C_2'(t) + \left[ \frac{1}{2}e^{-t}\csc t \right]e^t = 0 \quad \rightarrow \quad \frac{1}{2}\csc t + C_2'(t) + \frac{1}{2}\csc t = 0
\]
\[ C'_2(t) = -\csc t \]

Integrate both sides with respect to \( t \), setting the integration constant to zero.

\[ C_2(t) = \ln |\csc t + \cot t| \]

The particular solution is then

\[ y_p(t) = C_1(t)e^{-t} + C_2(t) + C_3(t)e^t \]

\[ = \left( \int^t \frac{1}{2} e^s \csc s \, ds \right) e^{-t} + \ln |\csc t + \cot t| + \left( \int^t \frac{1}{2} e^{-s} \csc s \, ds \right) e^t \]

\[ = \frac{1}{2} e^{-t} \int^t e^s \csc s \, ds + \ln |\csc t + \cot t| + \frac{1}{2} e^t \int^t e^{-s} \csc s \, ds, \]

and the general solution is

\[ y(t) = y_c(t) + y_p(t) \]

\[ = C_1 e^{-t} + C_2 + C_3 e^t + \frac{1}{2} e^{-t} \int^t e^s \csc s \, ds + \ln |\csc t + \cot t| + \frac{1}{2} e^t \int^t e^{-s} \csc s \, ds. \]

Differentiate it with respect to \( t \) twice.

\[ y'(t) = -C_1 e^{-t} + C_3 e^t - \frac{1}{2} e^{-t} \int^t e^s \csc s \, ds + \frac{1}{2} e^{-t}(e^t \csc t) - \csc t + \frac{1}{2} e^t \int^t e^{-s} \csc s \, ds + \frac{1}{2} e^t(e^{-t} \csc t) \]

\[ y''(t) = C_1 e^{-t} + C_3 e^t + \frac{1}{2} e^{-t} \int^t e^s \csc s \, ds - \frac{1}{2} e^{-t}(e^t \csc t) + \frac{1}{2} e^t \int^t e^{-s} \csc s \, ds + \frac{1}{2} e^t(e^{-t} \csc t) \]

Now apply the initial conditions to determine \( C_1, C_2, \) and \( C_3 \). Since they are given at \( t = \pi/2 \), let \( \pi/2 \) be the lower limit of integration in all integrals.

\[ y(\pi/2) = C_1 e^{-\pi/2} + C_2 + C_3 e^{\pi/2} = 2 \]

\[ y'(\pi/2) = -C_1 e^{-\pi/2} + C_3 e^{\pi/2} = 1 \]

\[ y''(\pi/2) = C_1 e^{-\pi/2} + C_3 e^{\pi/2} = -1 \]

Solving this system of equations yields \( C_1 = -e^{\pi/2}, C_2 = 3, \) and \( C_3 = 0. \)

\[ y(t) = -e^{\pi/2} e^{-t} + 3 + \frac{1}{2} e^{-t} \int_{\pi/2}^t e^s \csc s \, ds + \ln |\csc t + \cot t| + \frac{1}{2} e^t \int_{\pi/2}^t e^{-s} \csc s \, ds \]

\[ = 3 - e^{\pi/2}e^{-t} + \ln |\csc t + \cot t| + \frac{1}{2} \int_{\pi/2}^t (e^{-t+s} \csc s + e^{t-s} \csc s) \, ds \]

\[ = 3 - e^{\pi/2}e^{-t} + \ln |\csc t + \cot t| + \int_{\pi/2}^t \frac{e^{-(t-s)} + e^{t-s}}{2} \csc s \, ds \]

Therefore,

\[ y(t) = 3 - e^{\pi/2}e^{-t} + \ln |\csc t + \cot t| + \int_{\pi/2}^t \cosh(t - s) \csc s \, ds. \]