Problem 17

Find a formula involving integrals for a particular solution of the differential equation

\[ x^3 y''' - 3x^2 y'' + 6xy' - 6y = g(x), \quad x > 0. \]

*Hint:* Verify that \( x, x^2, \) and \( x^3 \) are solutions of the homogeneous equation.

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**Solution**

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of \( y_c(x) \) and \( y_p(x) \), the complementary solution and the particular solution, respectively.

\[ y(x) = y_c(x) + y_p(x) \]

The complementary solution satisfies the associated homogeneous equation.

\[ x^3 y_c''' - 3x^2 y_c'' + 6xy_c' - 6y_c = 0 \]  \hspace{1cm} (1)

Since this is an equidimensional equation, the solution is of the form \( y_c = x^r \).

\[ y_c = x^r \quad \rightarrow \quad y_c' = rx^{r-1} \quad \rightarrow \quad y_c'' = r(r-1)x^{r-2} \quad \rightarrow \quad y_c''' = r(r-1)(r-2)x^{r-3} \]

Substitute these expressions into the ODE.

\[ x^3r(r-1)(r-2)x^{r-3} - 3x^2r(r-1)x^{r-2} + 6xr^2x^{r-1} - 6x^r = 0 \]

\[ r(r-1)(r-2)x^r - 3r(r-1)x^r + 6rx^r - 6x^r = 0 \]

Divide both sides by \( x^r \).

\[ r(r-1)(r-2) - 3r(r-1) + 6r - 6 = 0 \]

\[ r^3 - 6r^2 + 11r - 6 = 0 \]

\[ (r - 1)(r - 2)(r - 3) = 0 \]

\[ r = \{1, 2, 3\} \]

Three solutions to equation (1) are then \( y_c = x^1 = x \) and \( y_c = x^2 \) and \( y_c = x^3 \). By the principle of superposition, the general solution for \( y_c \) is a linear combination of these three.

\[ y_c(x) = C_1x + C_2x^2 + C_3x^3 \]

On the other hand, the particular solution satisfies

\[ x^3y_p''' - 3x^2y_p'' + 6xy_p' - 6y_p = g(x). \] \hspace{1cm} (2)

According to the method of variation of parameters, the particular solution can be obtained by allowing the parameters in \( y_c(x) \) to vary.

\[ y_p(x) = C_1(x)x + C_2(x)x^2 + C_3(x)x^3 \]

Substitute this into equation (2).

\[ x^3[C_1(x)x + C_2(x)x^2 + C_3(x)x^3]''' - 3x^2[C_1(x)x + C_2(x)x^2 + C_3(x)x^3]'' \]

\[ + 6x[C_1(x)x + C_2(x)x^2 + C_3(x)x^3]' - 6[C_1(x)x + C_2(x)x^2 + C_3(x)x^3] = g(x) \]
Evaluate the derivatives.

\[ x^3[C_1'(x)x + C_1(x) + C_2'(x)x^2 + 2C_2(x)x + C_3'(x)x^3 + 3C_3(x)x^2]' = \]
\[ -3x^2[C_1'(x)x + C_1(x) + C_2'(x)x^2 + 2C_2(x)x + C_3'(x)x^3 + 3C_3(x)x^2]' \]
\[ + 6x[C_1'(x)x + C_1(x) + C_2'(x)x^2 + 2C_2(x)x + C_3'(x)x^3 + 3C_3(x)x^2] \]
\[ - 6[C_1(x)x + C_2(x)x^2 + C_3(x)x^3] = g(x) \]

If we set \( C_1'(x)x + C_2'(x)x^2 + C_3'(x)x^3 = 0 \), then this equation simplifies to

\[ x^3[C_1(x) + 2C_2(x)x + 3C_3(x)x^2]' = 3x^2[C_1(x) + 2C_2(x)x + 3C_3(x)x^2]' \]
\[ + 6x[C_1(x) + 2C_2(x)x + 3C_3(x)x^2] - 6[C_1(x)x + C_2(x)x^2 + C_3(x)x^3] = g(x) \]

\[ x^3[C_1'(x) + 2C_2'(x)x + 2C_2(x) + 3C_3'(x)x^2 + 6C_3(x)x]' = \]
\[ -3x^2[C_1'(x) + 2C_2'(x)x + 2C_2(x) + 3C_3'(x)x^2 + 6C_3(x)x] \]
\[ + 6x[C_1(x) + 2C_2(x)x + 3C_3(x)x^2] \]
\[ - 6[C_1(x)x + C_2(x)x^2 + C_3(x)x^3] = g(x). \]

If we set \( C_1'(x) + 2C_2'(x)x + 3C_3'(x)x^2 = 0 \), then this equation simplifies to

\[ x^3[2C_2(x) + 6C_3(x)x]' = 3x^2[2C_2(x) + 6C_3(x)x] \]
\[ + 6x[C_1(x) + 2C_2(x)x + 3C_3(x)x^2] - 6[C_1(x)x + C_2(x)x^2 + C_3(x)x^3] = g(x) \]

\[ x^3[2C_2'(x)x + 6C_3'(x)x + 6C_3(x)] - 3x^2[2C_2(x) + 6C_3(x)x] \]
\[ + 6x[C_1(x) + 2C_2(x)x + 3C_3(x)x^2] - 6[C_1(x)x + C_2(x)x^2 + C_3(x)x^3] = g(x) \]
\[ 2C_2'(x)x^3 + 6C_3'(x)x^4 = g(x). \]

As a result of using the method of variation of parameters, the problem of finding a particular solution has reduced to solving the following system of ODEs.

\[ C_1'(x)x + C_2'(x)x^2 + C_3'(x)x^3 = 0 \]  (3)
\[ C_1'(x) + 2C_2'(x)x + 3C_3'(x)x^2 = 0 \]  (4)
\[ 2C_2'(x)x^3 + 6C_3'(x)x^4 = g(x) \]  (5)

Multiply both sides of equation (4) by \( x \) and then subtract the respective sides of it from equation (3) to eliminate \( C_1'(x) \).

\[ -C_2'(x)x^2 - 2C_3'(x)x^3 = 0 \]

Solve this equation for \( C_2'(x) \)

\[ C_2'(x) = -2C_3'(x)x \]  (6)

and then plug it into equation (5).

\[ 2[2C_3'(x)x]x^3 + 6C_3'(x)x^4 = g(x) \]
\[ 2C_3'(x)x^4 = g(x) \]
\[ C_3'(x) = \frac{1}{2} x^{-4} g(x) \]

Integrate both sides with respect to \( x \), setting the integration constant to zero.

\[ C_3(x) = \int x^2 \frac{1}{2} s^{-4} g(s) \, ds \]

Substitute this result into equation (6) to get \( C_2(x) \).

\[ C_2'(x) = -2 \left[ \frac{1}{2} x^{-4} g(x) \right] x = -x^{-3} g(x) \]

Integrate both sides with respect to \( x \), setting the integration constant to zero.

\[ C_2(x) = \int x \frac{1}{2} s^{-3} g(s) \, ds \]

Substitute this result and the one for \( C_3(x) \) into equation (4) to get \( C_1(x) \).

\[ \begin{align*}
C_1'(x) + 2C_2'(x)x + 3C_3'(x)x^2 &= 0 \\
\Rightarrow \quad C_1'(x) + 2[-x^{-3} g(x)]x + 3 \left[ \frac{1}{2} x^{-4} g(x) \right] x^2 &= 0 \\
C_1'(x) - 2x^{-2} g(x) + \frac{3}{2} x^{-2} g(x) &= 0 \\
C_1'(x) &= \frac{1}{2} x^{-2} g(x)
\end{align*} \]

Integrate both sides with respect to \( x \), setting the integration constant to zero.

\[ C_1(x) = \int \frac{1}{2} s^{-2} g(s) \, ds \]

The particular solution is then

\[ y_p(x) = C_1(x)x + C_2(x)x^2 + C_3(x)x^3 \]

\[ \begin{align*}
&= \left[ \int \frac{1}{2} s^{-2} g(s) \, ds \right] x + \left[ \int \frac{1}{2} s^{-3} g(s) \, ds \right] x^2 + \left[ \int \frac{1}{2} s^{-4} g(s) \, ds \right] x^3 \\
&= \frac{1}{2} \int x s^{-2} g(s) \, ds - \frac{1}{2} \int x^2 s^{-3} g(s) \, ds + \frac{1}{2} \int x^3 s^{-4} g(s) \, ds \\
&= \frac{1}{2} \int x \left( \frac{x^2}{s^2} - \frac{2x^2}{s^3} + \frac{x^3}{s^4} \right) g(s) \, ds \\
&= \frac{1}{2} \int x \left( \frac{x^2}{s^2} - \frac{2xs}{s^3} + \frac{x^3}{s^4} \right) g(s) \, ds \\
&= \frac{1}{2} \int x \left( \frac{x^2}{s^2} - \frac{2xs}{s^3} + \frac{x^2}{s^4} \right) g(s) \, ds \\
&= \frac{1}{2} \int x \left( \frac{x^2}{s^2} - \frac{2xs}{s^3} + \frac{x^2}{s^4} \right) g(s) \, ds \\
&= \frac{x}{2} \int \frac{(x-s)^2}{s^4} g(s) \, ds \\
&= \frac{x}{2} \int \frac{(x-s)^2}{s^4} g(s) \, ds,
\end{align*} \]

and the general solution is

\[ y(x) = y_c(x) + y_p(x) \]

\[ = C_1 x + C_2 x^2 + C_3 x^3 + \frac{x}{2} \int \frac{(x-s)^2}{s^4} g(s) \, ds. \]