Problem 3

In each of Problems 1 through 6, use the method of variation of parameters to determine the general solution of the given differential equation.

\[ y''' - 2y'' - y' + 2y = e^{4t} \]

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of \( y_c(t) \) and \( y_p(t) \), the complementary solution and the particular solution, respectively.

\[ y(t) = y_c(t) + y_p(t) \]

The complementary solution satisfies the associated homogeneous equation.

\[ y'''_c - 2y''_c - y'_c + 2y_c = 0 \quad (1) \]

Since each term on the left has constant coefficients, the solution is of the form \( y_c = e^{rt} \).

\[
\begin{align*}
y_c &= e^{rt} &\rightarrow& & y'_c = re^{rt} &\rightarrow& & y''_c = r^2e^{rt} &\rightarrow& & y'''_c = r^3e^{rt}
\end{align*}
\]

Substitute these expressions into the ODE.

\[
r^3e^{rt} - 2(r^2e^{rt}) - re^{rt} + 2(e^{rt}) = 0
\]

Divide both sides by \( e^{rt} \).

\[
r^3 - 2r^2 - r + 2 = 0
\]

\[
(r - 2)(r^2 - 1) = 0
\]

\[
r = \{-1, 1, 2\}
\]

Three solutions to equation (1) are then \( y_c = e^{-t} \) and \( y_c = e^t \) and \( y_c = e^{2t} \). By the principle of superposition, the general solution for \( y_c \) is a linear combination of these three.

\[ y_c(t) = C_1e^{-t} + C_2e^t + C_3e^{2t} \]

On the other hand, the particular solution satisfies

\[ y'''_p - 2y''_p - y'_p + 2y_p = e^{4t} \quad (2) \]

According to the method of variation of parameters, the particular solution can be obtained by allowing the parameters in \( y_c(t) \) to vary.

\[ y_p(t) = C_1(t)e^{-t} + C_2(t)e^t + C_3(t)e^{2t} \]

Substitute this formula into equation (2).

\[
\begin{align*}
[C_1(t)e^{-t} + C_2(t)e^t + C_3(t)e^{2t}]''' &- 2[C_1(t)e^{-t} + C_2(t)e^t + C_3(t)e^{2t}]'' \\
&- [C_1(t)e^{-t} + C_2(t)e^t + C_3(t)e^{2t}]' + 2[C_1(t)e^{-t} + C_2(t)e^t + C_3(t)e^{2t}] = e^{4t}
\end{align*}
\]

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Evaluate the derivatives.

\[ [C_1'(t)e^{-t} - C_1(t)e^{-t} + C_2'(t)e^t + C_2(t)e^t + C_3'(t)e^{2t} + 2C_3(t)e^{2t}]^n \]
\[ -2[C_1'(t)e^{-t} - C_1(t)e^{-t} + C_2'(t)e^t + C_2(t)e^t + C_3'(t)e^{2t} + 2C_3(t)e^{2t}]' \]
\[ - [C_1'(t)e^{-t} - C_1(t)e^{-t} + C_2'(t)e^t + C_2(t)e^t + C_3'(t)e^{2t} + 2C_3(t)e^{2t}] \]
\[ + 2[C_1(t)e^{-t} + C_2(t)e^t + C_3(t)e^{2t}] = e^{4t} \]

If we set \( C_1'(t)e^{-t} + C_2'(t)e^t + C_3'(t)e^{2t} = 0 \), then this equation simplifies to

\[ [-C_1(t)e^{-t} + C_2(t)e^t + 2C_3(t)e^{2t}]'' - 2[-C_1(t)e^{-t} + C_2(t)e^t + 2C_3(t)e^{2t}]' \]
\[ - [-C_1(t)e^{-t} + C_2(t)e^t + 2C_3(t)e^{2t}] + 2[C_1(t)e^{-t} + C_2(t)e^t + C_3(t)e^{2t}] = e^{4t} \]

\[ [-C_1'(t)e^{-t} + C_2'(t)e^t + 2C_3'(t)e^{2t} + 4C_3(t)e^{2t}]' \]
\[ - 2[-C_1'(t)e^{-t} + C_2'(t)e^t + C_2'(t)e^t + C_2(t)e^t + 2C_3'(t)e^{2t} + 4C_3(t)e^{2t}]' \]
\[ - [-C_1'(t)e^{-t} + C_2'(t)e^t + 2C_3(t)e^{2t}] + 2[C_1(t)e^{-t} + C_2(t)e^t + C_3(t)e^{2t}] = e^{4t}. \]

If we set \( -C_1'(t)e^{-t} + C_2'(t)e^t + 2C_3'(t)e^{2t} = 0 \), then this equation simplifies to

\[ [C_1(t)e^{-t} + C_2(t)e^t + 4C_3(t)e^{2t}]' - 2[C_1(t)e^{-t} + C_2(t)e^t + 4C_3(t)e^{2t}] \]
\[ - [-C_1(t)e^{-t} + C_2(t)e^t + 2C_3(t)e^{2t}] + 2[C_1(t)e^{-t} + C_2(t)e^t + C_3(t)e^{2t}] = e^{4t} \]

\[ [C_1'(t)e^{-t} - C_2'(t)e^t + C_2(t)e^t + 4C_3'(t)e^{2t} + 8C_3(t)e^{2t}] - 2[C_1(t)e^{-t} + C_2(t)e^t + 4C_3(t)e^{2t}] \]
\[ - [-C_1(t)e^{-t} + C_2(t)e^t + 2C_3(t)e^{2t}] + 2[C_1(t)e^{-t} + C_2(t)e^t + C_3(t)e^{2t}] = e^{4t} \]
\[ C_1'(t)e^{-t} + C_2'(t)e^t + 4C_3'(t)e^{2t} = e^{4t}. \]

As a result of using the method of variation of parameters, the problem of finding a particular solution has reduced to solving the following system of ODEs.

\[ C_1'(t)e^{-t} + C_2'(t)e^t + C_3'(t)e^{2t} = 0 \] 
(3)
\[ -C_1'(t)e^{-t} + C_2'(t)e^t + 2C_3'(t)e^{2t} = 0 \] 
(4)
\[ C_1'(t)e^{-t} + C_2'(t)e^t + 4C_3'(t)e^{2t} = e^{4t} \] 
(5)

Multiply both sides of equation (3) by 2 and add the respective sides to equation (4) and subtract the respective sides from those of equation (5).

\[ 2[C_1'(t)e^{-t} + C_2'(t)e^t + C_3'(t)e^{2t}] + [-C_1'(t)e^{-t} + C_2'(t)e^t + 2C_3'(t)e^{2t}] - [C_1'(t)e^{-t} + C_2'(t)e^t + 4C_3'(t)e^{2t}] = -e^{4t} \]
\[ 2C_2'(t)e^t = -e^{4t} \]
\[ C_2'(t) = -\frac{1}{2}e^{4t} \]

Integrate both sides with respect to \( t \), setting the integration constant to zero.

\[ C_2(t) = -\frac{1}{6}e^{3t} \]

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Substitute this result back into equations (3) and (4).

\[ C_1'(t)e^{-t} + \left[ -\frac{1}{2}e^{3t} \right] e^t + C_3'(t)e^{2t} = 0 \]

\[-C_1'(t)e^{-t} + \left[ -\frac{1}{2}e^{3t} \right] e^t + 2C_3'(t)e^{2t} = 0 \]

Bring each second term to the right side and then add the respective sides of these equations.

\[ 3C_3'(t)e^{2t} = (e^{3t})e^t \]

\[ C_3'(t) = \frac{1}{3}e^{2t} \]

Integrate both sides with respect to \( t \), setting the integration constant to zero.

\[ C_3(t) = \frac{1}{6}e^{2t} \]

Now substitute this result and the one for \( C_4(t) \) into equation (3) to solve for \( C_1(t) \).

\[ C_1'(t)e^{-t} + C_2'(t)e^t + C_3'(t)e^{2t} = 0 \rightarrow C_1'(t)e^{-t} + \left( -\frac{1}{2}e^{3t} \right) e^t + \left( \frac{1}{3}e^{2t} \right) e^{2t} = 0 \]

\[ C_1'(t)e^{-t} - \frac{1}{2}e^{4t} + \frac{1}{3}e^{4t} = 0 \]

\[ C_1'(t)e^{-t} = \frac{1}{6}e^{4t} \]

\[ C_1'(t) = \frac{1}{6}e^{5t} \]

Integrate both sides with respect to \( t \), setting the integration constant to zero.

\[ C_1(t) = \frac{1}{30}e^{5t} \]

The particular solution is then

\[ y_p(t) = C_1(t)e^{-t} + C_2(t)e^t + C_3(t)e^{2t} \]

\[ = \left( \frac{1}{30}e^{5t} \right) e^{-t} + \left( -\frac{1}{6}e^{3t} \right) e^t + \left( \frac{1}{6}e^{2t} \right) e^{2t} \]

\[ = \frac{1}{30}e^{4t} - \frac{1}{6}e^{4t} + \frac{1}{6}e^{4t} \]

\[ = \frac{1}{30}e^{4t} \]

Therefore,

\[ y(t) = y_c(t) + y_p(t) \]

\[ = C_1e^{-t} + C_2e^t + C_3e^{2t} + \frac{1}{30}e^{4t} \]