Problem 7

In each of Problems 7 and 8, find the general solution of the given differential equation. Leave your answer in terms of one or more integrals.

\[ y''' - y'' + y' - y = \sec t, \quad -\pi/2 < t < \pi/2 \]

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as a sum of \( y_c(t) \) and \( y_p(t) \), the complementary solution and the particular solution, respectively.

\[ y(t) = y_c(t) + y_p(t) \]

The complementary solution satisfies the associated homogeneous equation.

\[ y'''_c - y''_c + y'_c - y_c = 0 \tag{1} \]

Since each term on the left has constant coefficients, the solution is of the form \( y_c = e^{rt} \).

\[ y_c = e^{rt} \rightarrow y'_c = re^{rt} \rightarrow y''_c = r^2e^{rt} \rightarrow y'''_c = r^3e^{rt} \]

Substitute these expressions into the ODE.

\[ r^3e^{rt} - r^2e^{rt} + re^{rt} - e^{rt} = 0 \]

Divide both sides by \( e^{rt} \).

\[ r^3 - r^2 + r - 1 = 0 \]

\[ (r - 1)(r^2 + 1) = 0 \]

\[ r = \{1, -i, i\} \]

Three solutions to equation (1) are then \( y_c = e^t \) and \( y_c = e^{-it} \) and \( y_c = e^{it} \). By the principle of superposition, the general solution for \( y_c \) is a linear combination of these three.

\[ y_c(t) = C_1e^t + C_2e^{-it} + C_3e^{it} \]

\[ = C_1e^t + C_2(\cos t - i \sin t) + C_3(\cos t + i \sin t) \]

\[ = C_1e^t + (C_2 + C_3) \cos t + (-iC_2 + iC_3) \sin t \]

\[ = C_1e^t + C_4 \cos t + C_5 \sin t \]

On the other hand, the particular solution satisfies

\[ y'''_p - y''_p + y'_p - y_p = \sec t. \tag{2} \]

According to the method of variation of parameters, the particular solution can be obtained by allowing the parameters in \( y_c(t) \) to vary.

\[ y_p(t) = C_1(t)e^t + C_4(t) \cos t + C_5(t) \sin t \]

Substitute this formula into equation (2).

\[ [C_1(t)e^t + C_4(t) \cos t + C_5(t) \sin t]''' - [C_1(t)e^t + C_4(t) \cos t + C_5(t) \sin t]'' \]

\[ + [C_1(t)e^t + C_4(t) \cos t + C_5(t) \sin t]' - [C_1(t)e^t + C_4(t) \cos t + C_5(t) \sin t] = \sec t \]

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Evaluate the derivatives.

\[ [C'_1(t)e^t + C_1(t)e^t + C'_4(t)\cos t - C_4(t)\sin t + C_5(t)\cos t]''
\[ - [C'_1(t)e^t + C_1(t)e^t + C'_4(t)\cos t - C_4(t)\sin t + C_5(t)\cos t]' + [C'_1(t)e^t + C_1(t)e^t + C'_4(t)\cos t - C_4(t)\sin t + C_5(t)\cos t] - [C'_1(t)e^t + C_4(t)\cos t + C_5(t)\sin t] = \sec t \]

If we set \( C'_1(t)e^t + C'_4(t)\cos t + C'_5(t)\sin t = 0 \), then this equation simplifies to

\[ [C_1(t)e^t - C_4(t)\sin t + C_5(t)\cos t]' - [C_1(t)e^t - C_4(t)\sin t + C_5(t)\cos t]' + [C_1(t)e^t - C_4(t)\sin t + C_5(t)\cos t] - [C_1(t)e^t + C_4(t)\cos t + C_5(t)\sin t] = \sec t \]

If we set \( C'_1(t)e^t - C'_4(t)\sin t + C'_5(t)\cos t = 0 \), then this equation simplifies to

\[ [C_1(t)e^t - C_4(t)\cos t - C_5(t)\sin t]' - [C_1(t)e^t - C_4(t)\cos t - C_5(t)\sin t]' + [C_1(t)e^t - C_4(t)\cos t - C_5(t)\sin t] - [C_1(t)e^t + C_4(t)\cos t + C_5(t)\sin t] = \sec t \]

As a result of using the method of variation of parameters, the problem of finding a particular solution has reduced to solving the following system of ODEs.

\[ C'_1(t)e^t + C_1(t)e^t + C'_4(t)\cos t - C_4(t)\sin t + C_5(t)\cos t = 0 \] (3)
\[ C'_1(t)e^t - C'_4(t)\sin t + C'_5(t)\cos t = 0 \] (4)
\[ C'_1(t)e^t - C'_4(t)\cos t - C'_5(t)\sin t = \sec t \] (5)

Add the respective sides of equations (3) and (5) together.

\[ 2C'_1(t)e^t = \sec t \]
\[ C'_1(t) = \frac{1}{2}e^{-t} \sec t \]

Integrate both sides with respect to \( t \), setting the integration constant to zero.

\[ C_1(t) = \int^t \frac{1}{2}e^{-s} \sec s \, ds \]
Subtract the respective sides of equation (4) from those of equation (3). Also, subtract the respective sides of equation (5) from those of equation (3).

\[ C_4'(t)(\cos t + \sin t) + C_5'(\sin t - \cos t) = 0 \]
\[ 2C_4'(t) \cos t + 2C_5'(t) \sin t = -\sec t \]

Solve this first equation for \( C_4'(t) \)

\[ C_4'(t) = \frac{\cos t - \sin t}{\cos t + \sin t} C_5'(t) \tag{6} \]

and then plug it into the second equation.

\[ 2 \left[ \frac{\cos t - \sin t}{\cos t + \sin t} C_5'(t) \right] \cos t + 2C_5'(t) \sin t = -\sec t \]

Divide both sides by 2 and factor \( C_5'(t) \).

\[ C_5'(t) \left[ \frac{\cos t - \sin t}{\cos t + \sin t} \cos t + \sin t + \frac{\sin t(\cos t + \sin t)}{\cos t + \sin t} \right] = -\frac{1}{2} \sec t \]
\[ C_5'(t) \left[ \frac{\cos^2 t - \sin^2 t + \sin t \cos t + \sin^2 t}{\cos t + \sin t} \right] = -\frac{1}{2} \sec t \]
\[ C_5'(t) \left( \frac{1}{\cos t + \sin t} \right) = -\frac{1}{2} \sec t \]
\[ C_5'(t) = -\frac{1}{2} \sec t(\cos t + \sin t) \]
\[ = -\frac{1}{2} - \frac{1}{2} \tan t \]

Integrate both sides with respect to \( t \), setting the integration constant to zero.

\[ C_5(t) = -\frac{t}{2} - \frac{1}{2} \ln |\sec t| \]
\[ = -\frac{t}{2} + \frac{1}{2} \ln |\cos t| \]

Because \(-\pi/2 < t < \pi/2\), the absolute value sign can be dropped. Now substitute the previous equation into equation (6).

\[ C_4'(t) = \frac{\cos t - \sin t}{\cos t + \sin t} \left[ -\frac{1}{2} \sec t(\cos t + \sin t) \right] \]
\[ = -\frac{1}{2} \sec t(\cos t - \sin t) \]
\[ = -\frac{1}{2} + \frac{1}{2} \tan t \]
Integrate both sides with respect to $t$, setting the integration constant to zero.

$$C_4(t) = -\frac{t}{2} + \frac{1}{2} \ln |\sec t|$$

$$= -\frac{t}{2} - \frac{1}{2} \ln |\cos t|$$

Because $-\pi/2 < t < \pi/2$, the absolute value sign can be dropped. The particular solution is then

$$y_p(t) = C_1(t)e^t + C_4(t) \cos t + C_5(t) \sin t$$

$$= \left[ \int \frac{1}{2} e^{-s} \sec s \, ds \right] e^t + \left[ -\frac{t}{2} - \frac{1}{2} \ln \cos t \right] \cos t + \left[ -\frac{t}{2} + \frac{1}{2} \ln \cos t \right] \sin t$$

$$= \left[ \int \frac{1}{2} e^{-s} \sec s \, ds - \frac{t}{2} \cos s - \frac{1}{2} \ln \cos t - \frac{t}{2} \sin t - \frac{1}{2} \sin t \ln \cos t \right]$$

$$= \frac{1}{2} \left[ \int e^{-s} \sec s \, ds - \frac{t}{2} (\sin t + \cos t) + \frac{1}{2} (\sin t - \cos t) \ln \cos t \right]$$

Therefore,

$$y(t) = y_c(t) + y_p(t)$$

$$= C_1 e^t + C_4 \cos t + C_5 \sin t + \frac{1}{2} \int e^{-s} \sec s \, ds - \frac{t}{2} (\sin t + \cos t) + \frac{1}{2} (\sin t - \cos t) \ln \cos t.$$