Problem 12

In each of Problems 9 through 16, determine the Taylor series about the point x_0 for the given function. Also determine the radius of convergence of the series.

$$x^2, \qquad x_0 = -1$$

Solution

The Taylor series expansion for a function $f(x)$ about the point $x = x_0$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Since $x_0 = -1$, this formula becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(-1)}{n!}(x + 1)^n.$$

The aim then is to determine the nth derivative of x^2 evaluated at $x = -1$. Start taking derivatives and try to observe a pattern.

$$\begin{aligned}
n = 0:&\quad f^{(0)}(x) = x^2 &\quad\to\quad& f^{(0)}(-1) = 1\\
n = 1:&\quad f^{(1)}(x) = 2x &\quad\to\quad& f^{(1)}(-1) = -2\\
n = 2:&\quad f^{(2)}(x) = 2 &\quad\to\quad& f^{(2)}(-1) = 2\\
n = 3:&\quad f^{(3)}(x) = 0 &\quad\to\quad& f^{(3)}(-1) = 0\\
n = 4:&\quad f^{(4)}(x) = 0 &\quad\to\quad& f^{(4)}(-1) = 0\\
&\quad\vdots
\end{aligned}$$

Notice that $f^{(n)}(-1) = 0$ for $n > 2$. Therefore,

$$\begin{aligned}
f(x) = x^2 &= \frac{f^{(0)}(-1)}{0!}(x+1)^0 + \frac{f^{(1)}(-1)}{1!}(x+1)^1 + \frac{f^{(2)}(-1)}{2!}(x+1)^2 + \sum_{n=3}^{\infty} \frac{f^{(n)}(-1)}{n!}(x+1)^n\\
&= (x+1)^0 - 2(x+1)^1 + (x+1)^2 + \sum_{n=2}^{\infty} \frac{0}{n!}(x+1)^n\\
&= 1 - 2(x+1) + (x+1)^2.
\end{aligned}$$

Since this series is finite, it is convergent for any value of x. That is, the radius of convergence is ∞.