Problem 16

In each of Problems 9 through 16, determine the Taylor series about the point x_0 for the given function. Also determine the radius of convergence of the series.

$$\frac{1}{1-x}, \qquad x_0 = 2$$

Solution

The Taylor series expansion for a function $f(x)$ about the point $x = x_0$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Since $x_0 = 2$, this formula becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x - 2)^n.$$

The aim then is to determine the nth derivative of $1/(1-x)$ evaluated at $x = 2$. Start taking derivatives and try to observe a pattern.

$$\begin{aligned}
n = 0: &\quad f^{(0)}(x) = (1-x)^{-1} &\quad\rightarrow\quad& f^{(0)}(2) = -1 \\
n = 1: &\quad f^{(1)}(x) = (1-x)^{-2} &\quad\rightarrow\quad& f^{(1)}(2) = 1 \\
n = 2: &\quad f^{(2)}(x) = 2(1-x)^{-3} &\quad\rightarrow\quad& f^{(2)}(2) = -2 \\
n = 3: &\quad f^{(3)}(x) = 6(1-x)^{-4} &\quad\rightarrow\quad& f^{(3)}(2) = 6 \\
n = 4: &\quad f^{(4)}(x) = 24(1-x)^{-5} &\quad\rightarrow\quad& f^{(4)}(2) = -24 \\
&\quad \vdots &&
\end{aligned}$$

Notice that $f^{(n)}(2) = (-1)^{n+1} n!$.

$$\begin{aligned}
f(x) = \frac{1}{1-x} &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} n!}{n!}(x-2)^n \\
&= \sum_{n=0}^{\infty} (-1)^{n+1}(x-2)^n
\end{aligned}$$

Apply the ratio test to determine the condition in which the series converges.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+2}(x-2)^{n+1}}{(-1)^{n+1}(x-2)^n} \right| \\
&= \lim_{n\to\infty} |(-1)(x-2)| \\
&= \lim_{n\to\infty} |x-2| \\
&= |x-2|
\end{aligned}$$

According to this test, the series is

$$\begin{cases} \text{convergent} & \text{if } |x-2| < 1 \\ \text{unknown} & \text{if } |x-2| = 1 \\ \text{divergent} & \text{if } |x-2| > 1 \end{cases}.$$

From the condition of convergence, which can also be written as $-1 < x - 2 < 1$, or $1 < x < 3$, we see that the center of convergence is at $x = 2$ and the radius of convergence is 1.