# Problem 26

In each of Problems 21 through 27, rewrite the given expression as a sum whose generic term involves $x^n$.

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n$$

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### Solution

Bring $x$ into the summand.

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1}$$

Write out the first term of the first sum.

$$a_1 + \sum_{n=2}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1}$$

Substitute $k = n - 2$ in the first sum and substitute $k = n$ in the second sum.

$$a_1 + \sum_{k+2=2}^{\infty} (k+2) a_{k+2} x^{k+1} + \sum_{k=0}^{\infty} a_k x^{k+1}$$

Solve for $k$.

$$a_1 + \sum_{k=0}^{\infty} (k+2) a_{k+2} x^{k+1} + \sum_{k=0}^{\infty} a_k x^{k+1}$$

Combine the sums.

$$a_1 + \sum_{k=0}^{\infty} [(k+2) a_{k+2} x^{k+1} + a_k x^{k+1}]$$

Factor $x^{k+1}$.

$$a_1 + \sum_{k=0}^{\infty} [(k+2) a_{k+2} + a_k] x^{k+1}$$

Substitute $m = k + 1$.

$$a_1 + \sum_{m-1=0}^{\infty} [(m+1) a_{m+1} + a_{m-1}] x^m$$

Solve for $m$.

$$a_1 + \sum_{m=1}^{\infty} [(m+1) a_{m+1} + a_{m-1}] x^m$$

As $m$ is only a dummy index, it can be replaced with $n$.

$$a_1 + \sum_{n=1}^{\infty} [(n+1) a_{n+1} + a_{n-1}] x^n$$