Problem 11

In each of Problems 9 through 16, determine the Taylor series about the point $x_0$ for the given function. Also determine the radius of convergence of the series.

$$x, \quad x_0 = 1$$

Solution

The Taylor series expansion for a function $f(x)$ about the point $x = x_0$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n.$$  

Since $x_0 = 1$, this formula becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (x - 1)^n.$$  

The aim then is to determine the $n$th derivative of $x$ evaluated at $x = 1$. Start taking derivatives and try to observe a pattern.

$n = 0$ : $f^{(0)}(x) = x \quad \rightarrow \quad f^{(0)}(1) = 1$  

$n = 1$ : $f^{(1)}(x) = 1 \quad \rightarrow \quad f^{(1)}(1) = 1$  

$n = 2$ : $f^{(2)}(x) = 0 \quad \rightarrow \quad f^{(2)}(1) = 0$  

$n = 3$ : $f^{(3)}(x) = 0 \quad \rightarrow \quad f^{(3)}(1) = 0$  

\vdots

Notice that $f^{(n)}(1) = 0$ for $n > 1$. Therefore,

$$f(x) = x = \frac{f^{(0)}(1)}{0!} (x - 1)^0 + \frac{f^{(1)}(1)}{1!} (x - 1)^1 + \sum_{n=2}^{\infty} \frac{f^{(n)}(1)}{n!} (x - 1)^n$$

$$= (x - 1)^0 + (x - 1)^1 + \sum_{n=2}^{\infty} \frac{0}{n!} (x - 1)^n$$

$$= 1 + (x - 1).$$

Since this series is finite, it is convergent for any value of $x$. That is, the radius of convergence is $\infty$. 

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