Problem 12

In each of Problems 9 through 16, determine the Taylor series about the point \( x_0 \) for the given function. Also determine the radius of convergence of the series.

\[ x^2, \quad x_0 = -1 \]

Solution

The Taylor series expansion for a function \( f(x) \) about the point \( x = x_0 \) is

\[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n. \]

Since \( x_0 = -1 \), this formula becomes

\[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(-1)}{n!} (x + 1)^n. \]

The aim then is to determine the \( n \)th derivative of \( x^2 \) evaluated at \( x = -1 \). Start taking derivatives and try to observe a pattern.

\begin{align*}
  n = 0 : \quad f^{(0)}(x) &= x^2 \quad \rightarrow \quad f^{(0)}(-1) = 1 \\
  n = 1 : \quad f^{(1)}(x) &= 2x \quad \rightarrow \quad f^{(1)}(-1) = -2 \\
  n = 2 : \quad f^{(2)}(x) &= 2 \quad \rightarrow \quad f^{(2)}(-1) = 2 \\
  n = 3 : \quad f^{(3)}(x) &= 0 \quad \rightarrow \quad f^{(3)}(-1) = 0 \\
  n = 4 : \quad f^{(4)}(x) &= 0 \quad \rightarrow \quad f^{(4)}(-1) = 0 \\
  \vdots
\end{align*}

Notice that \( f^{(n)}(-1) = 0 \) for \( n > 2 \). Therefore,

\[
  f(x) = x^2 = \frac{f^{(0)}(-1)}{0!} (x + 1)^0 + \frac{f^{(1)}(-1)}{1!} (x + 1)^1 + \frac{f^{(2)}(-1)}{2!} (x + 1)^2 + \sum_{n=3}^{\infty} \frac{f^{(n)}(-1)}{n!} (x + 1)^n
\]

\[
  = (x + 1)^0 - 2(x + 1)^1 + (x + 1)^2 + \sum_{n=2}^{\infty} 0 \frac{0}{n!}(x + 1)^n
\]

\[
  = 1 - 2(x + 1) + (x + 1)^2.
\]

Since this series is finite, it is convergent for any value of \( x \). That is, the radius of convergence is \( \infty \).