Problem 14

In each of Problems 9 through 16, determine the Taylor series about the point \(x_0\) for the given function. Also determine the radius of convergence of the series.

\[
\frac{1}{1 + x}, \quad x_0 = 0
\]

Solution

The Taylor series expansion for a function \(f(x)\) about the point \(x = x_0\) is

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n.
\]

Since \(x_0 = 0\), this formula reduces to

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.
\]

The aim then is to determine the \(n\)th derivative of \(1/(1 + x)\) evaluated at \(x = 0\). Start taking derivatives and try to observe a pattern.

\[
\begin{align*}
n = 0 : & \quad f^{(0)}(x) = (1 + x)^{-1} \quad \rightarrow \quad f^{(0)}(0) = 1 \\
n = 1 : & \quad f^{(1)}(x) = -(1 + x)^{-2} \quad \rightarrow \quad f^{(1)}(0) = -1 \\
n = 2 : & \quad f^{(2)}(x) = 2(1 + x)^{-3} \quad \rightarrow \quad f^{(2)}(0) = 2 \\
n = 3 : & \quad f^{(3)}(x) = -6(1 + x)^{-4} \quad \rightarrow \quad f^{(3)}(0) = -6 \\
n = 4 : & \quad f^{(4)}(x) = 24(1 + x)^{-5} \quad \rightarrow \quad f^{(4)}(0) = 24 \\
\end{align*}
\]

\[\vdots\]

Notice that \(f^{(n)}(0) = (-1)^n n!\).

\[
f(x) = \frac{1}{1 + x} = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!} x^n
\]

\[
= \sum_{n=0}^{\infty} (-1)^n x^n
\]

Apply the ratio test to determine the condition in which the series converges.

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{n+1} x^{n+1}}{(-1)^n x^n} \right|
\]

\[
= \lim_{n \to \infty} \left| (-1) x \right|
\]

\[
= \lim_{n \to \infty} \left| x \right|
\]

\[
= |x|
\]

www.stemjock.com
According to this test, the series is

\[
\begin{cases}
\text{convergent} & \text{if } |x| < 1 \\
\text{unknown} & \text{if } |x| = 1 \\
\text{divergent} & \text{if } |x| > 1
\end{cases}
\]

From the condition of convergence, which can also be written as \(-1 < x < 1\), we see that the center of convergence is at \(x = 0\) and the radius of convergence is 1.