Problem 15

In each of Problems 9 through 16, determine the Taylor series about the point \( x_0 \) for the given function. Also determine the radius of convergence of the series.

\[ \frac{1}{1-x}, \quad x_0 = 0 \]

Solution

The Taylor series expansion for a function \( f(x) \) about the point \( x = x_0 \) is

\[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n. \]

Since \( x_0 = 0 \), this formula reduces to

\[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n. \]

The aim then is to determine the \( n \)th derivative of \( 1/(1-x) \) evaluated at \( x = 0 \). Start taking derivatives and try to observe a pattern.

\[
\begin{align*}
 n = 0 : & \quad f^{(0)}(x) = (1-x)^{-1} \quad \rightarrow \quad f^{(0)}(0) = 1 \\
 n = 1 : & \quad f^{(1)}(x) = (1-x)^{-2} \quad \rightarrow \quad f^{(1)}(0) = 1 \\
 n = 2 : & \quad f^{(2)}(x) = 2(1-x)^{-3} \quad \rightarrow \quad f^{(2)}(0) = 2 \\
 n = 3 : & \quad f^{(3)}(x) = 6(1-x)^{-4} \quad \rightarrow \quad f^{(3)}(0) = 6 \\
 n = 4 : & \quad f^{(4)}(x) = 24(1-x)^{-5} \quad \rightarrow \quad f^{(4)}(0) = 24 \\
 \vdots
\end{align*}
\]

Notice that \( f^{(n)}(0) = n! \).

\[
\begin{align*}
 f(x) & = \frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{n!}{n!} x^n \\
 & = \sum_{n=0}^{\infty} x^n \\
 & = \sum_{n=0}^{\infty} x^n
\end{align*}
\]

Apply the ratio test to determine the condition in which the series converges.

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}}{x^n} \right| = \lim_{n \to \infty} |x| = |x|
\]

According to this test, the series is

\[
\begin{cases}
 \text{convergent} & \text{if } |x| < 1 \\
 \text{unknown} & \text{if } |x| = 1 \\
 \text{divergent} & \text{if } |x| > 1
\end{cases}
\]

From the condition of convergence, which can also be written as \(-1 < x < 1\), we see that the center of convergence is at \( x = 0 \) and the radius of convergence is 1.