Problem 26

In each of Problems 21 through 27, rewrite the given expression as a sum whose generic term involves $x^n$. 

$$ \sum_{n=1}^{\infty} na_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n $$

Solution

Bring $x$ into the summand. 

$$ \sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} $$

Write out the first term of the first sum. 

$$ a_1 + \sum_{n=2}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} $$

Substitute $k = n - 2$ in the first sum and substitute $k = n$ in the second sum. 

$$ a_1 + \sum_{k+2=2}^{\infty} (k + 2)a_{k+2} x^{k+1} + \sum_{k=0}^{\infty} a_k x^{k+1} $$

Solve for $k$. 

$$ a_1 + \sum_{k=0}^{\infty} [(k + 2)a_{k+2} x^{k+1} + a_k x^{k+1}] $$

Combine the sums. 

$$ a_1 + \sum_{k=0}^{\infty} [(k + 2)a_{k+2} + a_k] x^{k+1} $$

Factor $x^{k+1}$.

$$ a_1 + \sum_{k=0}^{\infty} [(k + 2)a_{k+2} + a_k] x^{k+1} $$

Substitute $m = k + 1$. 

$$ a_1 + \sum_{m=1}^{\infty} [(m + 1)a_{m+1} + a_{m-1}] x^m $$

Solve for $m$. 

$$ a_1 + \sum_{m=1}^{\infty} [(m + 1)a_{m+1} + a_{m-1}] x^m $$

As $m$ is only a dummy index, it can be replaced with $n$. 

$$ a_1 + \sum_{n=1}^{\infty} [(n + 1)a_{n+1} + a_{n-1}] x^n $$