Problem 27

In each of Problems 21 through 27, rewrite the given expression as a sum whose generic term involves $x^n$.

$$x \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n$$

Solution

Bring $x$ into the summand.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n$$

Because of the $n-1$ factor, the first sum can be started at $n = 1$.

$$\sum_{n=1}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n$$

Substitute $k = n - 1$ in the first sum and substitute $k = n$ in the second sum.

$$\sum_{k+1=1}^{\infty} (k+1)k a_{k+1} x^k + \sum_{k=0}^{\infty} a_k x^k$$

Solve for $k$.

$$\sum_{k=0}^{\infty} k(k+1) a_{k+1} x^k + \sum_{k=0}^{\infty} a_k x^k$$

Combine the two sums.

$$\sum_{k=0}^{\infty} [k(k+1) a_{k+1} x^k + a_k x^k]$$

Factor $x^k$.

$$\sum_{k=0}^{\infty} [k(k+1) a_{k+1} + a_k] x^k$$

As $k$ is only a dummy index, it can be replaced with $n$.

$$\sum_{n=0}^{\infty} [n(n+1) a_{n+1} + a_n] x^n$$