Problem 5

In each of Problems 1 through 8, determine the radius of convergence of the given power series.

\[ \sum_{n=1}^{\infty} \frac{(2x + 1)^n}{n^2} \]

Solution

Apply the ratio test to determine the condition in which the provided series converges.

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{|(2x+1)^{n+1}|}{(n+1)^2} \frac{n^2}{|2x+1|^n} = \lim_{n \to \infty} \left| \frac{n^2}{(n+1)^2} |2x+1| \right| = \lim_{n \to \infty} \left| \frac{1}{(1 + \frac{1}{n})^2} |2x+1| \right| = \lim_{n \to \infty} \left| \frac{1}{(1 + \frac{1}{n})^2} |2x+1| \right| = |2x+1|
\]

According to this test, the series is

\[
\begin{align*}
&\begin{cases}
\text{convergent} & \text{if } |2x + 1| < 1 \\
\text{unknown} & \text{if } |2x + 1| = 1 . \\
\text{divergent} & \text{if } |2x + 1| > 1
\end{cases}
\end{align*}
\]

From the condition of convergence, which can also be written as \(-1 < 2x + 1 < 1\), or \(-1 < x < 0\), we see that the center of convergence is at \(x = -1/2\) and the radius of convergence is 1/2.