Problem 7

In each of Problems 1 through 8, determine the radius of convergence of the given power series.

\[ \sum_{n=1}^{\infty} \frac{(-1)^n n^2 (x + 2)^n}{3^n} \]

Solution

Apply the ratio test to determine the condition in which the provided series converges.

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(-1)^{n+1} (n + 1)^2 (x + 2)^{n+1}}{(-1)^n n^2 (x + 2)^n} \frac{3^n}{3^{n+1}} \frac{1}{(x + 2)^n}
\]

\[
= \lim_{n \to \infty} \frac{(-1)^n (n + 1)^2}{n^2} \frac{3}{3} \left(\frac{1 + \frac{1}{n}}{3} (x + 2)\right)
\]

\[
= \lim_{n \to \infty} \frac{1}{3} \left(1 + \frac{1}{n}\right)^2 |x + 2|
\]

\[
= \frac{1}{3} |x + 2|
\]

According to this test, the series is

\[
\begin{cases} 
\text{convergent} & \text{if } \frac{1}{3} |x + 2| < 1 \\
\text{unknown} & \text{if } \frac{1}{3} |x + 2| = 1 \\
\text{divergent} & \text{if } \frac{1}{3} |x + 2| > 1
\end{cases}
\]

From the condition of convergence, which can also be written as \(|x + 2| < 3\), or \(-3 < x + 2 < 3\), we see that the center of convergence is at \(x = -2\) and the radius of convergence is 3.