Problem 9

In each of Problems 9 through 16, determine the Taylor series about the point \( x_0 \) for the given function. Also determine the radius of convergence of the series.

\[ \sin x, \quad x_0 = 0 \]

Solution

The Taylor series expansion for a function \( f(x) \) about the point \( x = x_0 \) is

\[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \]

Since \( x_0 = 0 \), this formula reduces to

\[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n. \]

The aim then is to determine the \( n \)th derivative of \( \sin x \) evaluated at \( x = 0 \). Start taking derivatives and try to observe a pattern.

\[
\begin{align*}
    n = 0: & \quad f^{(0)}(x) = \sin x \quad \rightarrow \quad f^{(0)}(0) = 0 \\
    n = 1: & \quad f^{(1)}(x) = \cos x \quad \rightarrow \quad f^{(1)}(0) = 1 \\
    n = 2: & \quad f^{(2)}(x) = -\sin x \quad \rightarrow \quad f^{(2)}(0) = 0 \\
    n = 3: & \quad f^{(3)}(x) = -\cos x \quad \rightarrow \quad f^{(3)}(0) = -1 \\
    n = 4: & \quad f^{(4)}(x) = \sin x \quad \rightarrow \quad f^{(4)}(0) = 0 \\
    n = 5: & \quad f^{(5)}(x) = \cos x \quad \rightarrow \quad f^{(5)}(0) = 1 \\
    \vdots
\end{align*}
\]

Since \( f^{(0)}(0) = 0 \), the sum can be started from \( n = 1 \).

\[ f(x) = \sin x = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \]

All the even values of \( n \) result in \( f^{(n)}(0) \) being zero, so sum over the odd integers only by making the substitution \( n = 2k + 1 \).

\[ \sin x = \sum_{2k+1=1}^{\infty} \frac{f^{(2k+1)}(0)}{(2k+1)!} x^{2k+1} \]

\[ = \sum_{k=0}^{\infty} \frac{f^{(2k+1)}(0)}{(2k+1)!} x^{2k+1} \]

Based on the results above, \( f^{(2k+1)}(0) = (-1)^k \). Therefore,

\[ \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}. \]
Apply the ratio test to determine the condition in which the series converges.

\[
\lim_{n \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \to \infty} \frac{(-1)^{k+1} x^{2(k+1)+1}}{(2k+3)!} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \\
= \lim_{k \to \infty} \frac{(-1)^{k+1} (2k+1)! x^{2k+3}}{(-1)^k (2k+3)! x^{2k+1}} \\
= \lim_{k \to \infty} \frac{1}{(2k+3)(2k+2)} x^2 \\
= 0 x^2
\]

According to this test, the series is

\[
\begin{cases} 
\text{convergent} & \text{if } 0 x^2 < 1 \\
\text{unknown} & \text{if } 0 x^2 = 1 \\
\text{divergent} & \text{if } 0 x^2 > 1
\end{cases}
\]

From the condition of convergence, which can also be written as \(|x| < \sqrt{1/0} = \infty\), or \(-\infty < x < \infty\), we see that the center of convergence is at \(x = 0\) and the radius of convergence is \(\infty\).