Problem 2

In each of Problems 1 through 14:

(a) Seek power series solutions of the given differential equation about the given point x_0; find the recurrence relation.

(b) Find the first four terms in each of two solutions y_1 and y_2 (unless the series terminates sooner).

(c) By evaluating the Wronskian $W(y_1, y_2)(x_0)$, show that y_1 and y_2 form a fundamental set of solutions.

(d) If possible, find the general term in each solution.

$$y'' - xy' - y = 0, \qquad x_0 = 0$$

Solution

$x = 0$ is not a zero of the coefficient of y'', so $x = 0$ is an ordinary point. As such, the solution for y can be represented as a power series centered at $x = 0$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate this series twice with respect to x to get y' and y''.

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these series into the ODE.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

Substitute $k = n - 2$ in the first sum, $k = n$ in the second sum, and $k = n$ in the third sum.

$$\sum_{k+2=2}^{\infty} (k+2)(k+1) a_{k+2} x^k - x \sum_{k=1}^{\infty} k a_k x^{k-1} - \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - x \sum_{k=1}^{\infty} k a_k x^{k-1} - \sum_{k=0}^{\infty} a_k x^k = 0$$

Bring x inside the summand and start the second series from $k = 0$.

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=0}^{\infty} k a_k x^k - \sum_{k=0}^{\infty} a_k x^k = 0$$

Since the three sums have the same limits and x^k, they can be combined.

$$\sum_{k=0}^{\infty} [(k+2)(k+1) a_{k+2} x^k - k a_k x^k - a_k x^k] = 0$$

Factor the summand.

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2}-(k+1)a_k]x^k=0$$

$$\sum_{k=0}^{\infty}(k+1)[(k+2)a_{k+2}-a_k]x^k=0$$

The quantity in square brackets must be zero.

$$(k+2)a_{k+2}-a_k=0$$

Solve for a_{k+2}.

$$a_{k+2}=\frac{a_k}{k+2}$$

Plug in different values of k to determine a pattern for a_k.

$$\begin{aligned}
a_2&=\frac{a_0}{2} & a_3&=\frac{a_1}{3}\\
a_4&=\frac{a_2}{4}=\frac{a_0}{4\cdot 2} & a_5&=\frac{a_3}{5}=\frac{a_1}{5\cdot 3}\\
a_6&=\frac{a_4}{6}=\frac{a_0}{6\cdot 4\cdot 2} & a_7&=\frac{a_5}{7}=\frac{a_1}{7\cdot 5\cdot 3}\\
&\vdots & &\vdots\\
a_{2k}&=\frac{a_0}{(2k)!!}=\frac{a_0}{2^k k!} & a_{2k+1}&=\frac{a_1}{(2k+1)!!}=\frac{a_1}{\frac{(2k+2)!}{2^{k+1}(k+1)!}}=\frac{2^{k+1}(k+1)!}{(2k+2)!}a_1
\end{aligned}$$

Therefore,

$$\begin{aligned}
y(x)&=\sum_{n=0}^{\infty}a_nx^n\\
&=\sum_{n\text{ even}}^{\infty}a_nx^n+\sum_{n\text{ odd}}^{\infty}a_nx^n\\
&=\sum_{k=0}^{\infty}a_{2k}x^{2k}+\sum_{k=0}^{\infty}a_{2k+1}x^{2k+1}\\
&=\sum_{k=0}^{\infty}\frac{a_0}{2^kk!}x^{2k}+\sum_{k=0}^{\infty}\frac{2^{k+1}(k+1)!}{(2k+2)!}a_1x^{2k+1}\\
&=a_0\sum_{k=0}^{\infty}\frac{x^{2k}}{2^kk!}+a_1\sum_{k=0}^{\infty}\frac{2^{k+1}(k+1)!}{(2k+2)!}x^{2k+1}\\
&=a_0\left(1+\frac{x^2}{2}+\frac{x^4}{8}+\frac{x^6}{48}+\cdots\right)+a_1\left(x+\frac{x^3}{3}+\frac{x^5}{15}+\frac{x^7}{105}+\cdots\right)\\
&=a_0y_1(x)+a_1y_2(x).
\end{aligned}$$

Now calculate the Wronskian of y_1 and y_2.

$$\begin{aligned}
W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\
&= y_1 y_2' - y_1' y_2 \\
&= \left(\sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!}\right) \left[\sum_{k=0}^{\infty} \frac{2^{k+1}(k+1)!}{(2k+2)!} x^{2k+1}\right]' - \left(\sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!}\right)' \left[\sum_{k=0}^{\infty} \frac{2^{k+1}(k+1)!}{(2k+2)!} x^{2k+1}\right] \\
&= \left(\sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!}\right) \left[\sum_{k=0}^{\infty} \frac{2^{k+1}(k+1)!}{(2k+2)!} (2k+1) x^{2k}\right] - \left(\sum_{k=1}^{\infty} 2k \frac{x^{2k-1}}{2^k k!}\right) \left[\sum_{k=0}^{\infty} \frac{2^{k+1}(k+1)!}{(2k+2)!} x^{2k+1}\right]
\end{aligned}$$

At $x = 0$ the Wronskian is

$$\begin{aligned}
W(y_1, y_2)(0) &= \left(\frac{1}{2^0 0!} + 0 + 0 + \cdots\right) \left[\frac{2^1 1!}{(2!)}(1) + 0 + 0 + \cdots\right] - (0 + 0 + \cdots)\,[0 + 0 + \cdots] \\
&= (1)(1) \\
&= 1.
\end{aligned}$$

Since the Wronskian is not zero at $x = 0$, the two functions, y_1 and y_2, form a fundamental set of solutions for the ODE.