Problem 11

In each of Problems 1 through 14:

(a) Seek power series solutions of the given differential equation about the given point x_0; find the recurrence relation.

(b) Find the first four terms in each of two solutions y_1 and y_2 (unless the series terminates sooner).

(c) By evaluating the Wronskian $W(y_1, y_2)(x_0)$, show that y_1 and y_2 form a fundamental set of solutions.

(d) If possible, find the general term in each solution.

$$(3 - x^2)y'' - 3xy' - y = 0, \qquad x_0 = 0$$

Solution

$x = 0$ is not a zero of the coefficient of y'', so $x = 0$ is an ordinary point. As such, the solution for y can be represented as a power series centered at $x = 0$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate this series twice with respect to x to get y' and y''.

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these series into the ODE.

$$(3 - x^2) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 3x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$3 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 3x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

Bring 3, x^2, and $3x$ into the respective summands.

$$\sum_{n=2}^{\infty} 3n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} 3n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

Start the second and third sums from $n = 0$. This can be done because of the n and $n - 1$ factors.

$$\sum_{n=2}^{\infty} 3n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1) a_n x^n - \sum_{n=0}^{\infty} 3n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

Substitute $k = n - 2$ in the first sum and $k = n$ in the others.

$$\sum_{k+2=2}^{\infty} 3(k+2)(k+1) a_{k+2} x^k - \sum_{k=0}^{\infty} k(k-1) a_k x^k - \sum_{k=0}^{\infty} 3k a_k x^k - \sum_{k=0}^{\infty} a_k x^k = 0$$

Solve for k.

$$\sum_{k=0}^{\infty} 3(k+2)(k+1)a_{k+2}x^k - \sum_{k=0}^{\infty} k(k-1)a_k x^k - \sum_{k=0}^{\infty} 3ka_k x^k - \sum_{k=0}^{\infty} a_k x^k = 0$$

Now that each of the sums has the same limits and factors of x, they can be combined.

$$\sum_{k=0}^{\infty} [3(k+2)(k+1)a_{k+2}x^k - k(k-1)a_k x^k - 3ka_k x^k - a_k x^k] = 0$$

Factor the summand.

$$\sum_{k=0}^{\infty} [3(k+2)(k+1)a_{k+2} - k(k-1)a_k - 3ka_k - a_k]x^k = 0$$

$$\sum_{k=0}^{\infty} [3(k+2)(k+1)a_{k+2} - (k^2+2k+1)a_k] = 0$$

$$\sum_{k=0}^{\infty} [3(k+2)(k+1)a_{k+2} - (k+1)^2 a_k] = 0$$

$$\sum_{k=0}^{\infty} (k+1)[3(k+2)a_{k+2} - (k+1)a_k] = 0$$

The coefficients must be zero.

$$3(k+2)a_{k+2} - (k+1)a_k = 0$$

Solve for a_{k+2}.

$$a_{k+2} = \frac{k+1}{3(k+2)}a_k$$

Plug in enough values of k to get four terms involving a_0 and four terms involving a_1.

$$\begin{aligned}
k=0: &\quad a_2 = \frac{1}{3(2)}a_0 = \frac{1}{3\cdot 2}a_0 \\
k=1: &\quad a_3 = \frac{2}{3(3)}a_1 = \frac{2}{3\cdot 3\cdot 1}a_1 \\
k=2: &\quad a_4 = \frac{3}{3(4)}a_2 = \frac{3}{3(4)}\left(\frac{1}{3\cdot 2}a_0\right) = \frac{3\cdot 1}{3^2\cdot 4\cdot 2}a_0 \\
k=3: &\quad a_5 = \frac{4}{3(5)}a_3 = \frac{4}{3(5)}\left(\frac{2}{3\cdot 3\cdot 1}a_1\right) = \frac{4\cdot 2}{3^2\cdot 5\cdot 3\cdot 1}a_1 \\
k=4: &\quad a_6 = \frac{5}{3(6)}a_4 = \frac{5}{3(6)}\left(\frac{3\cdot 1}{3^2\cdot 4\cdot 2}a_0\right) = \frac{5\cdot 3\cdot 1}{3^3\cdot 6\cdot 4\cdot 2}a_0 \\
k=5: &\quad a_7 = \frac{6}{3(7)}a_5 = \frac{6}{3(7)}\left(\frac{4\cdot 2}{3^2\cdot 5\cdot 3\cdot 1}a_1\right) = \frac{6\cdot 4\cdot 2}{3^3\cdot 7\cdot 5\cdot 3\cdot 1}a_1 \\
&\vdots
\end{aligned}$$

Therefore,

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n \text{ even}} a_n x^n + \sum_{n \text{ odd}} a_n x^n \\
&= \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1} \\
&= a_0 \sum_{k=0}^{\infty} \frac{(2k-1)!!}{3^k (2k)!!} x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{(2k)!!}{3^k (2k+1)!!} x^{2k+1} \\
&= a_0 \sum_{k=0}^{\infty} \frac{\frac{(2k)!}{2^k k!}}{3^k 2^k k!} x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{2^k k!}{3^k \frac{[2(k+1)]!}{2^{k+1}(k+1)!}} x^{2k+1} \\
&= a_0 \sum_{k=0}^{\infty} \frac{(2k)!}{12^k (k!)^2} x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{2^{2k+1}(k+1)!k!}{3^k (2k+2)!} x^{2k+1} \\
&= a_0 \sum_{k=0}^{\infty} \frac{(2k)!}{12^k (k!)^2} x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{2^{2k+1}(k+1)(k!)^2}{3^k (2k+2)(2k+1)!} x^{2k+1} \\
&= a_0 \sum_{k=0}^{\infty} \frac{(2k)!}{12^k (k!)^2} x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{2^{2k}(k!)^2}{3^k (2k+1)!} x^{2k+1} \\
&= a_0 \sum_{k=0}^{\infty} \frac{(2k)!}{12^k (k!)^2} x^{2k} + a_1 \sum_{k=0}^{\infty} \left(\frac{4}{3}\right)^k \frac{(k!)^2}{(2k+1)!} x^{2k+1} \\
&= a_0 \left(1 + \frac{x^2}{6} + \frac{x^4}{24} + \frac{5x^6}{432} + \cdots\right) + a_1 \left(x + \frac{2x^3}{9} + \frac{8x^5}{135} + \frac{16x^7}{945} + \cdots\right) \\
&= a_0 y_1(x) + a_1 y_2(x).
\end{aligned}$$

Now calculate the Wronskian of y_1 and y_2.

$$\begin{aligned}
W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\
&= y_1 y_2' - y_1' y_2 \\
&= \left(1 + \frac{x^2}{6} + \frac{x^4}{24} + \frac{5x^6}{432} + \cdots\right)\left(1 + \frac{2x^2}{3} + \frac{8x^4}{27} + \frac{16x^6}{135} + \cdots\right) \\
&\quad - \left(\frac{x}{3} + \frac{x^3}{6} + \frac{5x^5}{72} + \cdots\right)\left(x + \frac{2x^3}{9} + \frac{8x^5}{135} + \frac{16x^7}{945} + \cdots\right)
\end{aligned}$$

At $x = 0$ the Wronskian is nonzero,

$$W(y_1, y_2)(0) = (1)(1) - (0)(0) = 1,$$

which means that y_1 and y_2 form a fundamental set of solutions for the ODE.