Problem 20

Show directly, using the ratio test, that the two series solutions of Airy's equation about $x = 0$ converge for all x; see Eq. (20) of the text.

Solution

The Airy equation is

$$y'' - xy = 0.$$

$x = 0$ is not a zero of the coefficient of y'', so $x = 0$ is an ordinary point. As such, the solution for y can be represented as a power series centered at $x = 0$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate this series twice with respect to x to get y' and y''.

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these series into the ODE.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n = 0$$

Bring x into the respective summand.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Substitute $k + 1 = n - 2$ in the first sum and $k = n$ in the second sum.

$$\sum_{k+3=2}^{\infty} (k+3)(k+2) a_{k+3} x^{k+1} - \sum_{k=0}^{\infty} a_k x^{k+1} = 0$$

Solve for k.

$$\sum_{k=-1}^{\infty} (k+3)(k+2) a_{k+3} x^{k+1} - \sum_{k=0}^{\infty} a_k x^{k+1} = 0$$

Write out the first term of the first sum.

$$2a_2 + \sum_{k=0}^{\infty} (k+3)(k+2) a_{k+3} x^{k+1} - \sum_{k=0}^{\infty} a_k x^{k+1} = 0$$

Now that each of the sums has the same limits and factors of x, they can be combined.

$$2a_2 + \sum_{k=0}^{\infty} [(k+3)(k+2) a_{k+3} x^{k+1} - a_k x^{k+1}] = 0$$

Factor the summand.

$$2a_2 + \sum_{k=0}^{\infty} [(k+3)(k+2) a_{k+3} - a_k] x^{k+1} = 0 + 0x + 0x^2 + \cdots$$

Match the coefficients.

$$2a_2 = 0$$
$$(k+3)(k+2)a_{k+3} - a_k = 0$$

Solve for a_2 and a_{k+3}.

$$a_2 = 0$$
$$a_{k+3} = \frac{a_k}{(k+3)(k+2)}$$

Plug in enough values of k to find a pattern.

$$k=0: \quad a_3 = \frac{a_0}{(3)(2)} = \frac{a_0}{3\cdot 2} \qquad k=1: \quad a_4 = \frac{a_1}{(4)(3)} = \frac{a_1}{4\cdot 3} \qquad k=2: \quad a_5 = \frac{a_2}{(5)(4)} = 0$$
$$k=3: \quad a_6 = \frac{a_3}{(6)(5)} = \frac{a_0}{6\cdot 5\cdot 3\cdot 2} \qquad k=4: \quad a_7 = \frac{a_4}{(7)(6)} = \frac{a_1}{7\cdot 6\cdot 4\cdot 3} \qquad k=5: \quad a_8 = \frac{a_5}{(8)(7)} = 0$$
$$k=6: \quad a_9 = \frac{a_6}{(9)(8)} = \frac{a_0}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2} \qquad k=7: \quad a_{10} = \frac{a_7}{(10)(9)} = \frac{a_1}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3} \qquad k=8: \quad a_{11} = \frac{a_8}{(11)(10)} = 0$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

Generalize these results.

$$\begin{aligned} a_{3k} &= \frac{a_0}{[(3k)(3k-3)(3k-6)\cdots(3)][(3k-1)(3k-4)(3k-7)\cdots(2)]} \\ &= \frac{a_0}{[3^k(k)(k-1)(k-2)\cdots(1)]\left[3^k\left(k-\frac{1}{3}\right)\left(k-\frac{4}{3}\right)\left(k-\frac{7}{3}\right)\cdots\left(\frac{2}{3}\right)\right]} \\ &= \frac{a_0}{[3^k k!]\left[3^k\frac{\Gamma\left(k+\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\right]} \\ &= a_0\frac{\Gamma\left(\frac{2}{3}\right)}{3^{2k}k!\Gamma\left(k+\frac{2}{3}\right)} \end{aligned}$$

$$\begin{aligned}
a_{3k+1} &= \frac{a_1}{[(3k+1)(3k-2)(3k-5)\cdots(4)][(3k)(3k-3)(3k-6)\cdots(3)]} \\
&= \frac{a_1}{\left[3^k\left(k+\frac{1}{3}\right)\left(k-\frac{2}{3}\right)\left(k-\frac{5}{3}\right)\cdots\left(\frac{4}{3}\right)\right]\left[3^k(k)(k-1)(k-2)\cdots(1)\right]} \\
&= \frac{a_1}{\left[3^k\frac{\Gamma\left(k+\frac{4}{3}\right)}{\Gamma\left(\frac{4}{3}\right)}\right]\left[3^k k!\right]} \\
&= a_1\frac{\Gamma\left(\frac{4}{3}\right)}{3^{2k}k!\Gamma\left(k+\frac{4}{3}\right)} \\
a_{3k+2} &= 0
\end{aligned}$$

Therefore,

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{k=0}^{\infty} a_{3k}x^{3k} + \sum_{k=0}^{\infty} a_{3k+1}x^{3k+1} + \sum_{k=0}^{\infty} a_{3k+2}x^{3k+2} \\
&= a_0\sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{2}{3}\right)}{3^{2k}k!\Gamma\left(k+\frac{2}{3}\right)}x^{3k} + a_1\sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{4}{3}\right)}{3^{2k}k!\Gamma\left(k+\frac{4}{3}\right)}x^{3k+1}.
\end{aligned}$$

Now apply the ratio test to show that the first series solution converges.

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{A_{k+1}}{A_k}\right| &= \lim_{k\to\infty}\left|\frac{\frac{\Gamma\left(\frac{2}{3}\right)}{3^{2(k+1)}(k+1)!\Gamma\left(k+1+\frac{2}{3}\right)}x^{3(k+1)}}{\frac{\Gamma\left(\frac{2}{3}\right)}{3^{2k}k!\Gamma\left(k+\frac{2}{3}\right)}x^{3k}}\right| \\
&= \lim_{k\to\infty}\left|\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\frac{3^{2k}}{3^{2(k+1)}}\frac{k!}{(k+1)!}\frac{\Gamma\left(k+\frac{2}{3}\right)}{\Gamma\left(k+\frac{2}{3}+1\right)}\frac{x^{3(k+1)}}{x^{3k}}\right| \\
&= \lim_{k\to\infty}\left|\frac{1}{3^2}\frac{k!}{(k+1)k!}\frac{\Gamma\left(k+\frac{2}{3}\right)}{\left(k+\frac{2}{3}\right)\Gamma\left(k+\frac{2}{3}\right)}x^3\right| \\
&= \lim_{k\to\infty}\left|\frac{1}{9}\frac{1}{k+1}\frac{1}{k+\frac{2}{3}}x^3\right| \\
&= \lim_{k\to\infty}\frac{1}{9(k+1)\left(k+\frac{2}{3}\right)}|x^3| \\
&= 0|x^3|
\end{aligned}$$

According to this test, the first series is

$$\begin{cases} \text{convergent} & \text{if } 0|x^3| < 1 \\ \text{unknown} & \text{if } 0|x^3| = 1 \\ \text{divergent} & \text{if } 0|x^3| > 1 \end{cases}.$$

From the condition of convergence, which can also be written as $|x^3| < 1/0 = \infty$, or $-\infty < x^3 < \infty$, or $-\infty < x < \infty$, we see that the center of convergence is at $x = 0$ and the radius of convergence is ∞. Now apply the ratio test to the second series.

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{B_{k+1}}{B_k}\right| &= \lim_{k\to\infty}\left|\frac{\frac{\Gamma\left(\frac{4}{3}\right)}{3^{2(k+1)}(k+1)!\Gamma\left(k+1+\frac{4}{3}\right)}x^{3(k+1)+1}}{\frac{\Gamma\left(\frac{4}{3}\right)}{3^{2k}k!\Gamma\left(k+\frac{4}{3}\right)}x^{3k+1}}\right| \\
&= \lim_{k\to\infty}\left|\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{4}{3}\right)}\frac{3^{2k}}{3^{2(k+1)}}\frac{k!}{(k+1)!}\frac{\Gamma\left(k+\frac{4}{3}\right)}{\Gamma\left(k+\frac{4}{3}+1\right)}\frac{x^{3k+4}}{x^{3k+1}}\right| \\
&= \lim_{k\to\infty}\left|\frac{1}{3^2}\frac{k!}{(k+1)k!}\frac{\Gamma\left(k+\frac{4}{3}\right)}{\left(k+\frac{4}{3}\right)\Gamma\left(k+\frac{4}{3}\right)}x^3\right| \\
&= \lim_{k\to\infty}\left|\frac{1}{9}\frac{1}{k+1}\frac{1}{k+\frac{4}{3}}x^3\right| \\
&= \lim_{k\to\infty}\frac{1}{9(k+1)\left(k+\frac{4}{3}\right)}|x^3| \\
&= 0|x^3|
\end{aligned}$$

According to this test, the second series is

$$\begin{cases} \text{convergent} & \text{if } 0|x^3| < 1 \\ \text{unknown} & \text{if } 0|x^3| = 1 \\ \text{divergent} & \text{if } 0|x^3| > 1 \end{cases}.$$

From the condition of convergence, which can also be written as $|x^3| < 1/0 = \infty$, or $-\infty < x^3 < \infty$, or $-\infty < x < \infty$, we see that the center of convergence is at $x = 0$ and the radius of convergence is ∞.