Problem 21

The Hermite Equation. The equation

$$y'' - 2xy' + \lambda y = 0, \qquad -\infty < x < \infty,$$

where λ is a constant, is known as the Hermite[5] equation. It is an important equation in mathematical physics.

(a) Find the first four terms in each of two solutions about $x = 0$ and show that they form a fundamental set of solutions.

(b) Observe that if λ is a nonnegative even integer, then one or the other of the series solutions terminates and becomes a polynomial. Find the polynomial solutions for $\lambda = 0$, 2, 4, 6, 8, and 10. Note that each polynomial is determined only up to a multiplicative constant.

(c) The Hermite polynomial $H_n(x)$ is defined as the polynomial solution of the Hermite equation with $\lambda = 2n$ for which the coefficient of x^n is 2^n. Find $H_0(x), \ldots, H_5(x)$.

Solution

Part (a)

$x = 0$ is not a zero of the coefficient of y'', so $x = 0$ is an ordinary point. As such, the solution for y can be represented as a power series centered at $x = 0$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate this series twice with respect to x to get y' and y''.

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these series into the ODE.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2x \sum_{n=1}^{\infty} n a_n x^{n-1} + \lambda \sum_{n=0}^{\infty} a_n x^n = 0$$

Bring $2x$ and λ inside the respective summands.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} \lambda a_n x^n = 0$$

Because of the factor n, the second sum can be set to start from $n = 0$.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} \lambda a_n x^n = 0$$

[5] Charles Hermite (1822–1901) was an influential French analyst and algebraist. An inspiring teacher, he was professor at the École Polytechnique and the Sorbonne. He introduced the Hermite functions in 1864 and showed in 1873 that e is a transcendental number (that is, e is not a root of any polynomial equation with rational coefficients). His name is also associated with Hermitian matrices (see Section 7.3), some of whose properties he discovered.

Substitute $k = n - 2$ in the first sum and $k = n$ in the others.

$$\sum_{k+2=2}^{\infty} (k+2)(k+1)a_{k+2}x^k - \sum_{k=0}^{\infty} 2ka_kx^k + \sum_{k=0}^{\infty} \lambda a_kx^k = 0$$

Solve for k.

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k - \sum_{k=0}^{\infty} 2ka_kx^k + \sum_{k=0}^{\infty} \lambda a_kx^k = 0$$

Now that each of the sums has the same limits and factors of x, they can be combined.

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2}x^k - 2ka_kx^k + \lambda a_kx^k] = 0$$

Factor the summand.

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2} - 2ka_k + \lambda a_k]x^k = 0$$

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2} + (\lambda - 2k)a_k]x^k = 0$$

The coefficients must be zero.

$$(k+2)(k+1)a_{k+2} + (\lambda - 2k)a_k = 0$$

Solve for a_{k+2}.

$$a_{k+2} = \frac{2k - \lambda}{(k+2)(k+1)}a_k$$

Plug in enough values of k to get four terms involving a_0 and four terms involving a_1.

$$k = 0: \quad a_2 = \frac{0-\lambda}{(2)(1)}a_0 = \frac{0-\lambda}{2\cdot 1}a_0 \qquad\qquad k = 1: \quad a_3 = \frac{2-\lambda}{(3)(2)}a_1 = \frac{2-\lambda}{3\cdot 2}a_1$$

$$k = 2: \quad a_4 = \frac{4-\lambda}{(4)(3)}a_2 = \frac{(4-\lambda)(0-\lambda)}{4\cdot 3\cdot 2\cdot 1}a_0 \qquad\qquad k = 3: \quad a_5 = \frac{6-\lambda}{(5)(4)}a_3 = \frac{(6-\lambda)(2-\lambda)}{5\cdot 4\cdot 3\cdot 2}a_1$$

$$k = 4: \quad a_6 = \frac{8-\lambda}{(6)(5)}a_4 = \frac{(8-\lambda)(4-\lambda)(0-\lambda)}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}a_0 \qquad\qquad k = 5: \quad a_7 = \frac{10-\lambda}{(7)(6)}a_5 = \frac{(10-\lambda)(6-\lambda)(2-\lambda)}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}a_1$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

Therefore, the general solution to the Hermite equation is

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_nx^n \\
&= a_0 + a_1x + \frac{0-\lambda}{2\cdot 1}a_0x^2 + \frac{2-\lambda}{3\cdot 2}a_1x^3 + \frac{(4-\lambda)(0-\lambda)}{4\cdot 3\cdot 2\cdot 1}a_0x^4 + \frac{(6-\lambda)(2-\lambda)}{5\cdot 4\cdot 3\cdot 2}a_1x^5 \\
&\quad + \frac{(8-\lambda)(4-\lambda)(0-\lambda)}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}a_0x^6 + \frac{(10-\lambda)(6-\lambda)(2-\lambda)}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}a_1x^7 + \cdots \\
&= a_0\left[1 + \frac{0-\lambda}{2\cdot 1}x^2 + \frac{(4-\lambda)(0-\lambda)}{4\cdot 3\cdot 2\cdot 1}x^4 + \frac{(8-\lambda)(4-\lambda)(0-\lambda)}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}x^6 + \cdots\right] \\
&\quad + a_1\left[x + \frac{2-\lambda}{3\cdot 2}x^3 + \frac{(6-\lambda)(2-\lambda)}{5\cdot 4\cdot 3\cdot 2}x^5 + \frac{(10-\lambda)(6-\lambda)(2-\lambda)}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}x^7 + \cdots\right].
\end{aligned}$$

Denote these two series solutions as $y_1(x)$ and $y_2(x)$, respectively.

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

Calculate the Wronskian of y_1 and y_2.

$$\begin{aligned}
W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\
&= y_1 y_2' - y_1' y_2 \\
&= \left[1 + \frac{0-\lambda}{2\cdot 1}x^2 + \frac{(4-\lambda)(0-\lambda)}{4\cdot 3\cdot 2\cdot 1}x^4 + \frac{(8-\lambda)(4-\lambda)(0-\lambda)}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}x^6 + \cdots\right] \\
&\qquad \times \left[1 + \frac{2-\lambda}{2}x^2 + \frac{(6-\lambda)(2-\lambda)}{4\cdot 3\cdot 2}x^4 + \frac{(10-\lambda)(6-\lambda)(2-\lambda)}{6\cdot 5\cdot 4\cdot 3\cdot 2}x + \cdots\right] \\
&\qquad\qquad - \left[\frac{0-\lambda}{1}x + \frac{(4-\lambda)(0-\lambda)}{3\cdot 2\cdot 1}x^3 + \frac{(8-\lambda)(4-\lambda)(0-\lambda)}{5\cdot 4\cdot 3\cdot 2\cdot 1}x^5 + \cdots\right] \\
&\qquad\qquad\qquad \times \left[x + \frac{2-\lambda}{3\cdot 2}x^3 + \frac{(6-\lambda)(2-\lambda)}{5\cdot 4\cdot 3\cdot 2}x^5 + \frac{(10-\lambda)(6-\lambda)(2-\lambda)}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}x^7 + \cdots\right]
\end{aligned}$$

At $x = 0$ the Wronskian is nonzero,

$$W(y_1, y_2)(0) = (1)(1) - (0)(0) = 1,$$

which means y_1 and y_2 form a fundamental set of solutions for the ODE.

Part (b)

If $\lambda = 0$, then the first series solution terminates.

$$y_1(x) = 1$$

If $\lambda = 2$, then the second series solution terminates.

$$y_2(x) = x$$

If $\lambda = 4$, then the first series solution terminates.

$$y_1(x) = 1 - 2x^2$$

If $\lambda = 6$, then the second series solution terminates.

$$y_2(x) = x - \frac{2}{3}x^3$$

If $\lambda = 8$, then the first series solution terminates.

$$y_1(x) = 1 - 4x^2 + \frac{4}{3}x^4$$

If $\lambda = 10$, then the second series solution terminates.

$$y_2(x) = x - \frac{4}{3}x^3 + \frac{4}{15}x^5$$

Part (c)

The first five Hermite polynomials are obtained from these previous polynomial solutions. Write them so that the coefficients of x^n are 2^n, where $\lambda = 2n$.

$$
\begin{aligned}
n = 0: &\quad y_1 = 1 = 2^0 x^0 && \to \quad H_0(x) = 1 \\
n = 1: &\quad y_2 = x = \frac{1}{2}(2^1 x^1) && \to \quad H_1(x) = 2x \\
n = 2: &\quad y_1 = 1 - 2x^2 = -\frac{1}{2}(2^2 x^2 - 2) && \to \quad H_2(x) = 4x^2 - 2 \\
n = 3: &\quad y_2 = x - \frac{2}{3}x^3 = -\frac{1}{12}(2^3 x^3 - 12x) && \to \quad H_3(x) = 8x^3 - 12x \\
n = 4: &\quad y_1 = 1 - 4x^2 + \frac{4}{3}x^4 = \frac{1}{12}(12 - 48x^2 + 2^4 x^4) && \to \quad H_4(x) = 16x^4 - 48x^2 + 12 \\
n = 5: &\quad y_2 = x - \frac{4}{3}x^3 + \frac{4}{15}x^5 = \frac{1}{120}(120x - 160x^3 + 2^5 x^5) && \to \quad H_5(x) = 32x^5 - 160x^3 + 120x
\end{aligned}
$$