Problem 22

Consider the initial value problem \( y' = \sqrt{1 - y^2}, \ y(0) = 0. \)

(a) Show that \( y = \sin x \) is the solution of this initial value problem.

(b) Look for a solution of the initial value problem in the form of a power series about \( x = 0. \)

Find the coefficients up to the term in \( x^3 \) in this series.

Solution

\[
\frac{dy}{dx} = \sqrt{1 - y^2}
\]

Separate variables.

\[
\frac{dy}{\sqrt{1 - y^2}} = dx
\]

Integrate both sides.

\[
\int \frac{dy}{\sqrt{1 - y^2}} = \int dx
\]

To evaluate the integral on the left, make the substitution \( y = \sin \theta. \) Then \( dy = \cos \theta \, d\theta. \)

\[
\int \frac{\cos \theta \, d\theta}{\sqrt{1 - \sin^2 \theta}} = \int dx
\]

\[
\int \frac{\cos \theta \, d\theta}{\cos^2 \theta} = \int dx
\]

\[
\int \frac{\cos \theta \, d\theta}{\cos \theta} = \int dx
\]

\[
\int d\theta = \int dx
\]

\[
\theta = x + C
\]

\[
\sin^{-1} y = x + C
\]

Apply the initial condition \( y(0) = 0 \) to determine \( C. \)

\[
0 = C
\]

Therefore,

\[
\sin^{-1} y = x
\]

\[
y(x) = \sin x.
\]

The series solution for the ODE is just the Taylor series expansion of \( \sin x \) about \( x = 0. \)

\[
y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{6} + \cdots
\]