Problem 23

In each of Problems 23 through 28, plot several partial sums in a series solution of the given initial value problem about $x = 0$, thereby obtaining graphs analogous to those in Figures 5.2.1 through 5.2.4.

$$y'' - xy' - y = 0, \qquad y(0) = 1, \quad y'(0) = 0; \qquad \text{see Problem 2}$$

Solution

In Problem 2 the general solution was found to be

$$\begin{aligned} y(x) &= a_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!} + a_1 \sum_{k=0}^{\infty} \frac{2^{k+1}(k+1)!}{(2k+2)!} x^{2k+1} \\ &= a_0 \left(1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + \cdots\right) + a_1 \left(x + \frac{x^3}{3} + \frac{x^5}{15} + \frac{x^7}{105} + \cdots\right). \end{aligned}$$

Differentiate it with respect to x.

$$y'(x) = a_0 \left(x + \frac{x^3}{2} + \frac{x^5}{8} + \cdots\right) + a_1 \left(1 + x^2 + \frac{x^4}{3} + \frac{x^6}{15} + \cdots\right)$$

Now apply the initial conditions, $y(0) = 1$ and $y'(0) = 0$, to determine a_0 and a_1.

$$\begin{aligned} y(0) &= a_0 = 1 \\ y'(0) &= a_1 = 0 \end{aligned}$$

Therefore,

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!} = \sum_{k=0}^{\infty} \frac{(x^2/2)^k}{k!} = e^{x^2/2} \\ &= 1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + \cdots. \end{aligned}$$

Below is a plot of the various partial sums versus x.