Problem 28

In each of Problems 23 through 28, plot several partial sums in a series solution of the given initial value problem about $x = 0$, thereby obtaining graphs analogous to those in Figures 5.2.1 through 5.2.4.

$$(1-x)y'' + xy' - 2y = 0, \qquad y(0) = 0, \quad y'(0) = 1$$

Solution

$x = 0$ is an ordinary point, so the solution can be represented as a power series.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate this series twice with respect to x to get y' and y''.

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these series into the ODE.

$$(1-x)\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^n = 0$$

Bring x, x, and 2 into the respective summands.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^{n} - \sum_{n=0}^{\infty} 2a_n x^n = 0$$

Because of the $n-1$ factor, the second sum can be started from $n = 1$. Similarly, because of the n factor, the third sum can be started from $n = 0$.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=0}^{\infty} n a_n x^{n} - \sum_{n=0}^{\infty} 2a_n x^n = 0$$

Substitute $k = n - 2$ in the first sum, $k = n - 1$ in the second sum, and $k = n$ in the others.

$$\sum_{k+2=2}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k+1=1}^{\infty} (k+1)k a_{k+1} x^k + \sum_{k=0}^{\infty} k a_k x^k - \sum_{k=0}^{\infty} 2a_k x^k = 0$$

Solve for k.

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=0}^{\infty} k(k+1) a_{k+1} x^k + \sum_{k=0}^{\infty} k a_k x^k - \sum_{k=0}^{\infty} 2a_k x^k = 0$$

Now that each of the sums has the same limits and factors of x, they can be combined.

$$\sum_{k=0}^{\infty} [(k+2)(k+1) a_{k+2} x^k - k(k+1) a_{k+1} x^k + k a_k x^k - 2a_k x^k] = 0$$

Factor the summand.

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2} - k(k+1)a_{k+1} + ka_k - 2a_k]x^k = 0$$

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2} - k(k+1)a_{k+1} + (k-2)a_k]x^k = 0$$

The coefficients must be zero.

$$(k+2)(k+1)a_{k+2} - k(k+1)a_{k+1} + (k-2)a_k = 0$$

Solve for a_{k+2}.

$$a_{k+2} = \frac{k(k+1)a_{k+1} - (k-2)a_k}{(k+2)(k+1)}$$

Plug in enough values of k to get four terms involving a_0 and four terms involving a_1.

$$\begin{aligned}
k=0: \quad & a_2 = \frac{0(1)a_1 - (-2)a_0}{(2)(1)} = \frac{2a_0}{2\cdot 1} = a_0 \\
k=1: \quad & a_3 = \frac{1(2)a_2 - (-1)a_1}{(3)(2)} = \frac{1}{3}\left(\frac{a_0}{2\cdot 1}\right) = \frac{a_0}{3} + \frac{a_1}{6} \\
k=2: \quad & a_4 = \frac{2(3)a_3 - (0)a_2}{(4)(3)} = \frac{a_3}{2} = \frac{a_0}{6} + \frac{a_1}{12} \\
k=3: \quad & a_5 = \frac{3(4)a_4 - (1)a_3}{(5)(4)} = \frac{a_0}{12} + \frac{a_1}{24} \\
k=4: \quad & a_6 = \frac{4(5)a_5 - (2)a_4}{(6)(5)} = \frac{2a_0}{45} + \frac{a_1}{45} \\
& \vdots
\end{aligned}$$

As a result, the general solution is

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + a_0 x^2 + \left(\frac{a_0}{3} + \frac{a_1}{6}\right)x^3 + \left(\frac{a_0}{6} + \frac{a_1}{12}\right)x^4 + \left(\frac{a_0}{12} + \frac{a_1}{24}\right)x^5 + \left(\frac{2a_0}{45} + \frac{a_1}{45}\right)x^6 + \cdots \\
&= a_0\left(1 + x^2 + \frac{x^3}{3} + \frac{x^4}{6} + \cdots\right) + a_1\left(x + \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{24} + \cdots\right).
\end{aligned}$$

Differentiate it with respect to x.

$$y'(x) = a_0\left(2x + x^2 + \frac{2x^3}{3} + \cdots\right) + a_1\left(1 + \frac{x^2}{2} + \frac{x^3}{3} + \frac{5x^4}{24} + \cdots\right)$$

Now apply the initial conditions, $y(0) = 0$ and $y'(0) = 1$, to determine a_0 and a_1.

$$y(0) = a_0 = 0$$
$$y'(0) = a_1 = 1$$

Therefore,

$$y(x) = x + \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{24} + \cdots.$$

Below is a plot of the various partial sums versus x.