Problem 12

In each of Problems 1 through 14:

(a) Seek power series solutions of the given differential equation about the given point $x_0$; find the recurrence relation.

(b) Find the first four terms in each of two solutions $y_1$ and $y_2$ (unless the series terminates sooner).

(c) By evaluating the Wronskian $W(y_1, y_2)(x_0)$, show that $y_1$ and $y_2$ form a fundamental set of solutions.

(d) If possible, find the general term in each solution.

$$(1 - x)y'' + xy' - y = 0, \quad x_0 = 0$$

Solution

$x = 0$ is not a zero of the coefficient of $y''$, so $x = 0$ is an ordinary point. As such, the solution for $y$ can be represented as a power series centered at $x = 0$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate this series twice with respect to $x$ to get $y'$ and $y''$.

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

Substitute these series into the ODE.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

Bring $x$ and $x$ into the respective summands.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

Because of the $n - 1$ factor, the second sum can be started from $n = 1$. Similarly, because of the $n$ factor, the third sum can be started from $n = 0$.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

Substitute $k = n - 2$ in the first sum, $k = n - 1$ in the second sum, and $k = n$ in the others.

$$\sum_{k+2=2}^{\infty} (k + 2)(k + 1)a_{k+2} x^k - \sum_{k+1=1}^{\infty} (k + 1)k a_{k+1} x^k + \sum_{k=0}^{\infty} k a_k x^k - \sum_{k=0}^{\infty} a_k x^k = 0$$
Solve for $k$.
\[ \sum_{k=0}^{\infty} \left( (k+2)(k+1)a_{k+2}x^k - \sum_{k=0}^{\infty} k(k+1)a_{k+1}x^k + \sum_{k=0}^{\infty} ka_kx^k - \sum_{k=0}^{\infty} a_kx^k \right) = 0 \]

Now that each of the sums has the same limits and factors of $x$, they can be combined.
\[ \sum_{k=0}^{\infty} \left[ (k+2)(k+1)a_{k+2}x^k - k(k+1)a_{k+1}x^k + k a_kx^k - a_kx^k \right] = 0 \]

Factor the summand.
\[ \sum_{k=0}^{\infty} \left[ (k+2)(k+1)a_{k+2}x^k - k(k+1)a_{k+1}x^k + (k-1)a_kx^k \right] = 0 \]

The coefficients must be zero.
\[ (k+2)(k+1)a_{k+2} - k(k+1)a_{k+1} + (k-1)a_k = 0 \]

Solve for $a_{k+2}$.
\[ a_{k+2} = \frac{k(k+1)a_{k+1} - (k-1)a_k}{(k+2)(k+1)} \]

Plug in enough values of $k$ to get four terms involving $a_0$ and four terms involving $a_1$.
\[ \begin{align*}
  k = 0 : & \quad a_2 = \frac{0(1)a_1 - (-1)a_0}{(2)(1)} = \frac{a_0}{2\cdot 1} \\
  k = 1 : & \quad a_3 = \frac{1(2)a_2 - 0}{(3)(2)} = \frac{1}{3} \left( \frac{a_0}{2\cdot 1} \right) = \frac{a_0}{3\cdot 2\cdot 1} \\
  k = 2 : & \quad a_4 = \frac{2(3)a_3 - a_2}{(4)(3)} = \frac{2}{4} \left( \frac{a_0}{3\cdot 2\cdot 1} \right) - \frac{1}{4\cdot 3} \left( \frac{a_0}{2\cdot 1} \right) = \frac{a_0}{4\cdot 3\cdot 2\cdot 1}
\end{align*} \]

Therefore,
\[ y(x) = \sum_{n=0}^{\infty} a_nx^n \]
\[ = a_0 + a_1x + \frac{a_0}{2\cdot 1}x^2 + \frac{a_0}{3\cdot 2\cdot 1}x^3 + \frac{a_0}{4\cdot 3\cdot 2\cdot 1}x^4 + \cdots \]
\[ = a_1x + a_0 \left( 1 + \frac{x^2}{2\cdot 1} + \frac{x^3}{3\cdot 2\cdot 1} + \frac{x^4}{4\cdot 3\cdot 2\cdot 1} + \cdots \right) \]
\[ = a_1x + a_0 \left( -x + 1 + \frac{x^2}{1!} + \frac{x^3}{2\cdot 1} + \frac{x^4}{3\cdot 2\cdot 1} + \frac{x^5}{4\cdot 3\cdot 2\cdot 1} + \cdots \right) \]
\[ = a_1x + a_0 \left( -x + \sum_{n=0}^{\infty} \frac{x^n}{n!} \right) \]
\[ = a_1x + a_0(-x + e^x) \]
\[ = ay_2(x) + a_0y_1(x). \]
Now calculate the Wronskian of \( y_1 \) and \( y_2 \).

\[
W(y_1, y_2) = \begin{vmatrix}
  y_1 & y_2 \\
  y'_1 & y'_2
\end{vmatrix} \\
= y_1 y'_2 - y'_1 y_2 \\
= (-x + e^x)(1) - (-1 + e^x)(x)
\]

At \( x = 0 \) the Wronskian is nonzero,

\[
W(y_1, y_2)(0) = (1)(1) - (0)(0) = 1,
\]

which means that \( y_1 \) and \( y_2 \) form a fundamental set of solutions for the ODE.