Problem 15

In each of Problems 15 through 18:

(a) Find the first five nonzero terms in the solution of the given initial value problem.

(b) Plot the four-term and the five-term approximations to the solution on the same axes.

(c) From the plot in part (b) estimate the interval in which the four-term approximation is reasonably accurate.

\[ y'' - xy' - y = 0, \quad y(0) = 2, \quad y'(0) = 1; \quad \text{see Problem 2} \]

Solution

The initial conditions are provided at \( x = 0 \), which is not a zero of the coefficient of \( y'' \); thus, it is an ordinary point. As such, the solution for \( y \) can be represented as a power series centered at \( x = 0 \).

\[ y(x) = \sum_{n=0}^{\infty} a_n x^n \]

Differentiate this series twice with respect to \( x \) to get \( y' \) and \( y'' \).

\[ y = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \Rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \]

Substitute these series into the ODE.

\[ \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0 \]

Substitute \( k = n - 2 \) in the first sum, \( k = n \) in the second sum, and \( k = n \) in the third sum.

\[ \sum_{k+2=2}^{\infty} (k+2)(k+1) a_{k+2} x^k - x \sum_{k=1}^{\infty} k a_k x^{k-1} - \sum_{k=0}^{\infty} a_k x^k = 0 \]

\[ \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - x \sum_{k=1}^{\infty} k a_k x^{k-1} - \sum_{k=0}^{\infty} a_k x^k = 0 \]

Bring \( x \) inside the summand and start the second series from \( k = 0 \).

\[ \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=0}^{\infty} k a_k x^k - \sum_{k=0}^{\infty} a_k x^k = 0 \]

Since the three sums have the same limits and \( x^k \), they can be combined.

\[ \sum_{k=0}^{\infty} [(k+2)(k+1)a_{k+2}x^k - ka_kx^k - a_kx^k] = 0 \]

Factor the summand.

\[ \sum_{k=0}^{\infty} [(k+2)(k+1)a_{k+2} - (k+1)a_k]x^k = 0 \]

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\[
\sum_{k=0}^{\infty} (k+1)[(k+2)a_{k+2} - a_k]x^k = 0
\]

The quantity in square brackets must be zero.

\[(k+2)a_{k+2} - a_k = 0\]

Solve for \(a_{k+2}\).

\[a_{k+2} = \frac{a_k}{k+2}\]

Plug in different values of \(k\) to determine a pattern for \(a_k\).

\[
\begin{align*}
a_2 &= \frac{a_0}{2} \quad a_3 = \frac{a_1}{3} \\
a_4 &= \frac{a_2}{4} = \frac{a_0}{4 \cdot 2} \quad a_5 = \frac{a_3}{5} = \frac{a_1}{5 \cdot 3} \\
a_6 &= \frac{a_4}{6} = \frac{a_0}{6 \cdot 4 \cdot 2} \quad a_7 = \frac{a_5}{7} = \frac{a_1}{7 \cdot 5 \cdot 3} \\
\vdots
\end{align*}
\]

\[
\begin{align*}
a_{2k} &= \frac{a_0}{(2k)!!} = \frac{a_0}{2^k k!} \\
a_{2k+1} &= \frac{a_1}{(2k+1)!!} = \frac{a_1}{2^{k+1}(k+1)!!} = \frac{2^{k+1}(k+1)!}{(2k+2)!}a_1
\end{align*}
\]

As a result, the general solution is

\[
y(x) = \sum_{n=0}^{\infty} a_n x^n
\]

\[
= \sum_{n \text{ even}} a_n x^n + \sum_{n \text{ odd}} a_n x^n
\]

\[
= \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}
\]

\[
= \sum_{k=0}^{\infty} \frac{a_0}{2^k k!} x^{2k} + \sum_{k=0}^{\infty} \frac{2^{k+1}(k+1)!}{(2k+2)!}a_1 x^{2k+1}
\]

\[
= a_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!} + a_1 \sum_{k=0}^{\infty} \frac{2^{k+1}(k+1)!}{(2k+2)!} x^{2k+1}
\]

\[
= a_0 \left(1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + \cdots \right) + a_1 \left( x + \frac{x^3}{3} + \frac{x^5}{15} + \frac{x^7}{105} + \cdots \right).
\]

Differentiate it with respect to \(x\).

\[
y'(x) = a_0 \sum_{k=1}^{\infty} \frac{x^{2k-1}}{2^{k-1}(k-1)!} + a_1 \sum_{k=0}^{\infty} \frac{2^{k+1}(k+1)!}{(2k+2)!} (2k+1)x^{2k}
\]

Apply the initial conditions now to determine \(a_0\) and \(a_1\).

\[
y(0) = a_0 = 2
\]

\[
y'(0) = a_1 = 1
\]
Therefore,

\[ y(x) = 2 \left( 1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + \cdots \right) + \left( x + \frac{x^3}{3} + \frac{x^5}{15} + \frac{x^7}{105} + \cdots \right) \]

\[ = 2 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{4} + \cdots. \]

Below is a plot of the four-term approximation superimposed on the plot of the five-term approximation.

The interval in which the four-term approximation is accurate is roughly \(-0.75 < x < 0.75\)—the two curves start to deviate outside of it.