Problem 17

In each of Problems 15 through 18:

(a) Find the first five nonzero terms in the solution of the given initial value problem.

(b) Plot the four-term and the five-term approximations to the solution on the same axes.

(c) From the plot in part (b) estimate the interval in which the four-term approximation is reasonably accurate.

\[ y'' + xy' + 2y = 0, \quad y(0) = 4, \quad y'(0) = -1; \text{ see Problem 7} \]

Solution

The initial conditions are provided at \( x = 0 \), which is not a zero of the coefficient of \( y'' \); thus, it is an ordinary point. As such, the solution for \( y \) can be represented as a power series centered at \( x = 0 \).

\[ y(x) = \sum_{n=0}^{\infty} a_n x^n \]

Differentiate this series twice with respect to \( x \) to get \( y' \) and \( y'' \).

\[ y = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=1}^{\infty} na_n x^{n-1} \quad \rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \]

Substitute these series into the ODE.

\[ \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x \sum_{n=1}^{\infty} na_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0 \]

Bring \( x \) and 2 into the respective summands.

\[ \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0 \]

Substitute \( k = n - 2 \) in the first sum and \( k = n \) in the other sums.

\[ \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k + \sum_{k=1}^{\infty} ka_k x^k + \sum_{k=0}^{\infty} 2a_k x^k = 0 \]

Solve for \( k \) in the first sum. The second sum can be set to start from \( k = 0 \) because \( k \) is present in the summand.

\[ \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k + \sum_{k=0}^{\infty} ka_k x^k + \sum_{k=0}^{\infty} 2a_k x^k = 0 \]

Now that each of the sums has the same limits and factors of \( x \), they can be combined.

\[ \sum_{k=0}^{\infty} [(k+2)(k+1)a_{k+2} x^k + ka_k x^k + 2a_k x^k] = 0 \]

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Factor the summand.

\[
\sum_{k=0}^{\infty} [(k+2)(k+1)a_{k+2} + ka_k + 2a_k]x^k = 0
\]

\[
\sum_{k=0}^{\infty} [(k+2)(k+1)a_{k+2} + (k+2)a_k]x^k = 0
\]

\[
\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} + a_k]x^k = 0
\]

The coefficients must be zero.

\[(k+1)a_{k+2} + a_k = 0\]

Solve for \(a_{k+2}\).

\[a_{k+2} = -\frac{a_k}{k+1}\]

Plug in enough values of \(k\) to get four terms involving \(a_0\) and four terms involving \(a_1\).

\[k = 0: \quad a_2 = -\frac{a_0}{1}\]

\[k = 1: \quad a_3 = -\frac{a_1}{2}\]

\[k = 2: \quad a_4 = -\frac{a_2}{3} = -\frac{a_0}{3 \cdot 1}\]

\[k = 3: \quad a_5 = -\frac{a_3}{4} = -\frac{a_1}{4 \cdot 2}\]

\[k = 4: \quad a_6 = -\frac{a_4}{5} = -\frac{a_0}{5 \cdot 3 \cdot 1}\]

\[k = 5: \quad a_7 = -\frac{a_5}{6} = -\frac{a_1}{6 \cdot 4 \cdot 2}\]

\[\vdots\]

As a result, the general solution is

\[y(x) = \sum_{n=0}^{\infty} a_n x^n\]

\[= a_0 + a_1 x - \frac{a_0 x^2}{1} - \frac{a_1 x^3}{2} + \frac{a_0 x^4}{3 \cdot 1} + \frac{a_1 x^5}{4 \cdot 2} - \frac{a_0 x^6}{5 \cdot 3 \cdot 1} - \frac{a_1 x^7}{6 \cdot 4 \cdot 2} + \cdots\]

\[= a_0 \left(1 - \frac{x^2}{1} + \frac{x^4}{3 \cdot 1} - \frac{x^6}{5 \cdot 3 \cdot 1} + \cdots\right) + a_1 \left(x - \frac{x^3}{2} + \frac{x^5}{4 \cdot 2} - \frac{x^7}{6 \cdot 4 \cdot 2} + \cdots\right)\]

\[= a_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n-1)!} + a_1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!!}\]

\[= a_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2^n n!}\]

\[= a_0 \sum_{n=0}^{\infty} (-1)^n \frac{2^n n!}{(2n)!} x^{2n} + a_1 \sum_{n=0}^{\infty} (-1)^n \frac{2^n n!}{2^n n!} x^{2n+1}\]

Differentiate it with respect to \(x\).

\[y'(x) = a_0 \sum_{n=1}^{\infty} (-1)^n \frac{2^n n!}{(2n)!} (2n) x^{2n-1} + a_1 \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1) x^{2n}}{2^n n!}\]
Apply the initial conditions now to determine $a_0$ and $a_1$.

\[
y(0) = a_0 = 4 \\
y'(0) = a_1 = -1
\]

Therefore,

\[
y(x) = 4 \sum_{n=0}^{\infty} (-1)^n \frac{2^n n!}{(2n)!} x^{2n} - \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2^n n!} \\
= 4 \left( 1 - x^2 + \frac{x^4}{3} - \frac{x^6}{15} + \cdots \right) - \left( x - \frac{x^3}{2} + \frac{x^5}{8} - \frac{x^7}{48} + \cdots \right) \\
= 4 - x - 4x^2 + \frac{x^3}{2} + \frac{4x^4}{3} - \cdots.
\]

Below is a plot of the four-term approximation superimposed on the plot of the five-term approximation.

The interval in which the four-term approximation is accurate is roughly $-0.5 < x < 0.5$—the two curves start to deviate outside of it.