Problem 19

(a) By making the change of variable \( x - 1 = t \) and assuming that \( y \) has a Taylor series in powers of \( t \), find two series solutions of

\[
y'' + (x - 1)^2 y' + (x^2 - 1)y = 0
\]

in powers of \( x - 1 \).

(b) Show that you obtain the same result by assuming that \( y \) has a Taylor series in powers of \( x - 1 \) and also expressing the coefficient \( x^2 - 1 \) in powers of \( x - 1 \).

Solution

Part (a)

Rewrite the ODE.

\[
\frac{d^2 y}{dx^2} + (x - 1)^2 \frac{dy}{dx} + (x - 1)(x + 1)y = 0
\]

Make the change of variables, \( t = x - 1 \).

\[
\frac{d^2 y}{dx^2} + t^2 \frac{dy}{dx} + t(t + 2)y = 0
\]

The aim now is to find what \( dy/dt \) and \( d^2 y/dx^2 \) are in terms of this new variable. Use the chain rule.

\[
\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = \frac{dy}{dx}(1) = \frac{dy}{dx},
\]

\[
\frac{d^2 y}{dt^2} = \frac{d}{dt} \left( \frac{dy}{dt} \right) = \frac{dx}{dt} \frac{d}{dx} \left( \frac{dy}{dt} \right) = (1) \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d^2 y}{dx^2}
\]

As a result of changing variables, the new ODE is

\[
\frac{d^2 y}{dt^2} + t^2 \frac{dy}{dt} + t(t + 2)y = 0.
\]

Assume that \( y \) has a Taylor series in powers of \( t \).

\[
y(t) = \sum_{n=0}^{\infty} a_n t^n
\]

Differentiate this series twice with respect to \( t \) to get \( dy/dt \) and \( d^2 y/dt^2 \).

\[
y = \sum_{n=0}^{\infty} a_n t^n \quad \rightarrow \quad \frac{dy}{dt} = \sum_{n=1}^{\infty} a_n n t^{n-1} \quad \rightarrow \quad \frac{d^2 y}{dt^2} = \sum_{n=2}^{\infty} a_n n(n - 1) t^{n-2}
\]

Substitute these series into the ODE.

\[
\sum_{n=2}^{\infty} a_n n(n - 1) t^{n-2} + t^2 \sum_{n=1}^{\infty} a_n n t^{n-1} + t(t + 2) \sum_{n=0}^{\infty} a_n t^n = 0
\]

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\[
\begin{align*}
\sum_{n=2}^{\infty} a_n n(n-1) t^{n-2} + t^2 \sum_{n=1}^{\infty} a_n n t^{n-1} + (t^2 + 2t) \sum_{n=0}^{\infty} a_n t^n &= 0 \\
\sum_{n=2}^{\infty} a_n n(n-1) t^{n-2} + t^2 \sum_{n=1}^{\infty} a_n n t^{n-1} + t^2 \sum_{n=0}^{\infty} a_n t^n + 2t \sum_{n=0}^{\infty} a_n t^n &= 0
\end{align*}
\]

Bring \( t^2, t^2, \) and \( 2t \) into the respective summands.

\[
\begin{align*}
\sum_{n=2}^{\infty} a_n n(n-1) t^{n-2} + \sum_{n=1}^{\infty} a_n n t^{n-1} + \sum_{n=0}^{\infty} a_n t^n + \sum_{n=0}^{\infty} 2a_n t^{n+1} &= 0 \\
\end{align*}
\]

Substitute \( k + 2 = n - 2 \) in the first sum, \( k + 2 = n + 1 \) in the second sum, \( k = n \) in the third sum, and \( k + 2 = n + 1 \) in the fourth sum.

\[
\begin{align*}
\sum_{k=4}^{\infty} a_{k+4}(k+4)(k+3) t^{k+2} + \sum_{k+1=1}^{\infty} a_{k+1}(k + 1) t^{k+2} + \sum_{k=0}^{\infty} a_k t^{k+2} + \sum_{k=0}^{\infty} 2a_{k+1} t^{k+2} &= 0
\end{align*}
\]

Solve for \( k \).

\[
\begin{align*}
\sum_{k=-2}^{\infty} (k+4)(k+3)a_{k+4} t^{k+2} + \sum_{k=0}^{\infty} (k+1)a_{k+1} t^{k+2} + \sum_{k=0}^{\infty} a_k t^{k+2} + \sum_{k=0}^{\infty} 2a_{k+1} t^{k+2} &= 0
\end{align*}
\]

Write out the first two terms of the first sum and the first term of the fourth sum.

\[
\begin{align*}
2a_2 + 6a_3 t + \sum_{k=0}^{\infty} (k+4)(k+3)a_{k+4} t^{k+2} + \sum_{k=0}^{\infty} (k+1)a_{k+1} t^{k+2} + \sum_{k=0}^{\infty} a_k t^{k+2} + 2a_0 t + \sum_{k=0}^{\infty} 2a_{k+1} t^{k+2} &= 0
\end{align*}
\]

Now that each of the sums has the same limits and factors of \( t \), they can be combined.

\[
\begin{align*}
2a_2 + (2a_0 + 6a_3) t + \sum_{k=0}^{\infty} [(k+4)(k+3)a_{k+4} t^{k+2} + (k+1)a_{k+1} t^{k+2} + a_k t^{k+2} + 2a_{k+1} t^{k+2}] &= 0
\end{align*}
\]

Factor the summand.

\[
\begin{align*}
2a_2 + (2a_0 + 6a_3) t + \sum_{k=0}^{\infty} [(k+4)(k+3)a_{k+4} + (k+1)a_{k+1} + a_k + 2a_{k+1}] t^{k+2} &= 0
\end{align*}
\]

\[
\begin{align*}
2a_2 + (2a_0 + 6a_3) t + \sum_{k=0}^{\infty} [(k+4)(k+3)a_{k+4} + (k+3)a_{k+1} + a_k] t^{k+2} &= 0 + 0t + 0t^2 + \cdots
\end{align*}
\]

Match the coefficients on both sides.

\[
\begin{align*}
2a_2 &= 0 \\
2a_0 + 6a_3 &= 0 \\
(k + 4)(k + 3)a_{k+4} + (k + 3)a_{k+1} + a_k &= 0
\end{align*}
\]

Solve for \( a_2, a_3, \) and \( a_{k+4} \).

\[
\begin{align*}
a_2 &= 0 \\
(a_3 &= -\frac{a_0}{3} \\
a_{k+4} &= -\frac{(k + 3)a_{k+1} + a_k}{(k + 4)(k + 3)}
\end{align*}
\]

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Plug in enough values of $k$ to get four terms involving $a_0$ and four terms involving $a_1$.

\[
\begin{align*}
k = 0 : & \quad a_4 = -\frac{3a_1 + a_0}{4 \cdot 3} = -\frac{a_0}{12} - \frac{a_1}{4} \\
k = 1 : & \quad a_5 = -\frac{4a_2 + a_1}{5 \cdot 4} = -\frac{a_1}{20} \\
k = 2 : & \quad a_6 = -\frac{5a_3 + a_2}{6 \cdot 5} = \frac{a_0}{18} \\
k = 3 : & \quad a_7 = -\frac{6a_4 + a_3}{7 \cdot 6} = \frac{a_0}{252} + \frac{a_1}{28}
\end{align*}
\]

Therefore,

\[
y(t) = \sum_{n=0}^{\infty} a_n t^n \\
= a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots \\
= a_0 + a_1 t - \frac{a_0}{3} t^3 + \left( -\frac{a_0}{12} - \frac{a_1}{4} \right) t^4 - \frac{a_1}{20} t^5 + \frac{a_0}{18} t^6 + \left( \frac{a_0}{252} + \frac{a_1}{28} \right) t^7 + \cdots \\
= a_0 \left( 1 - \frac{t^3}{3} - \frac{t^4}{12} + \frac{t^6}{18} + \cdots \right) + a_1 \left( t - \frac{t^4}{4} - \frac{t^5}{20} + \frac{t^7}{28} + \cdots \right).
\]

Changing back to the original variable $x$, we have

\[
y(x) = a_0 \left[ 1 - \frac{1}{3} (x - 1)^3 - \frac{1}{12} (x - 1)^4 + \frac{1}{18} (x - 1)^6 + \cdots \right] \\
+ a_1 \left[ (x - 1) - \frac{1}{4} (x - 1)^4 - \frac{1}{20} (x - 1)^5 + \frac{1}{28} (x - 1)^7 + \cdots \right] \\
= a_0 y_1(x) + a_1 y_2(x).
\]
Part (b)

Rewrite the ODE.

\[ \frac{d^2y}{dx^2} + (x-1)^2 \frac{dy}{dx} + (x-1)(x+1)y = 0 \]
\[ \frac{d^2y}{dx^2} + (x-1)^2 \frac{dy}{dx} + (x-1)(x-1+2)y = 0 \]
\[ \frac{d^2y}{dx^2} + (x-1)^2 \frac{dy}{dx} + (x-1)^2y + 2(x-1)y = 0 \]

Assume that \( y \) has a Taylor series expansion in powers of \( x-1 \).

\[ y(x) = \sum_{n=0}^{\infty} a_n(x-1)^n \]

Differentiate this series twice with respect to \( x \) to get \( dy/dx \) and \( d^2y/dx^2 \).

\[ y = \sum_{n=0}^{\infty} a_n(x-1)^n \quad \rightarrow \quad \frac{dy}{dx} = \sum_{n=1}^{\infty} na_n(x-1)^{n-1} \quad \rightarrow \quad \frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} \]

Substitute these expressions into the ODE.

\[ \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + (x-1)^2 \sum_{n=1}^{\infty} na_n(x-1)^{n-1} \]
\[ + (x-1)^2 \sum_{n=0}^{\infty} a_n(x-1)^n + 2(x-1) \sum_{n=0}^{\infty} a_n(x-1)^n = 0 \]

Bring \( (x-1)^2 \), \( (x-1)^2 \), and \( 2(x-1) \) into the respective summands.

\[ \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=1}^{\infty} na_n(x-1)^{n+1} + \sum_{n=0}^{\infty} a_n(x-1)^{n+2} + \sum_{n=0}^{\infty} 2a_n(x-1)^{n+1} = 0 \]

Substitute \( k+2 = n-2 \) in the first sum, \( k+2 = n+1 \) in the second sum, \( k = n \) in the third sum, and \( k+2 = n+1 \) in the fourth sum.

\[ \sum_{k+4=2}^{\infty} a_{k+4}(k+4)(k+3)(x-1)^{k+2} + \sum_{k+1=1}^{\infty} a_{k+1}(k+1)(x-1)^{k+2} + \sum_{k=0}^{\infty} a_{k}(x-1)^{k+2} + \sum_{k+1=0}^{\infty} 2a_{k+1}(x-1)^{k+2} = 0 \]

Solve for \( k \).

\[ \sum_{k=-2}^{\infty} (k+4)(k+3)a_{k+4}(x-1)^{k+2} + \sum_{k=0}^{\infty} (k+1)a_{k+1}(x-1)^{k+2} + \sum_{k=0}^{\infty} a_{k}(x-1)^{k+2} + \sum_{k=-1}^{\infty} 2a_{k+1}(x-1)^{k+2} = 0 \]

Write out the first two terms of the first sum and the first term of the fourth sum.

\[ 2a_2 + 6a_3(x-1) + \sum_{k=0}^{\infty} (k+4)(k+3)a_{k+4}(x-1)^{k+2} + \sum_{k=0}^{\infty} (k+1)a_{k+1}(x-1)^{k+2} \]
\[ + \sum_{k=0}^{\infty} a_{k}(x-1)^{k+2} + 2a_0(x-1) + \sum_{k=0}^{\infty} 2a_{k+1}(x-1)^{k+2} = 0 \]
Now that each of the sums has the same limits and factors of \((x - 1)\), they can be combined.

\[
2a_2 + (2a_0 + 6a_3)(x - 1) + \sum_{k=0}^{\infty} [(k + 4)(k + 3)a_{k+4}(x - 1)^{k+2} + (k + 1)a_{k+1}(x - 1)^{k+2} + a_k(x - 1)^{k+2} + 2a_{k+1}(x - 1)^{k+2}] = 0
\]

Factor the summand.

\[
2a_2 + (2a_0 + 6a_3)(x - 1) + \sum_{k=0}^{\infty} [(k + 4)(k + 3)a_{k+4} + (k + 1)a_{k+1} + a_k + 2a_{k+1}](x - 1)^{k+2} = 0
\]

\[
2a_2 + (2a_0 + 6a_3)(x - 1) + \sum_{k=0}^{\infty} [(k + 4)(k + 3)a_{k+4} + (k + 3)a_{k+1} + a_k](x - 1)^{k+2} = 0 + 0(x - 1) + 0(x - 1)^2 + \cdots
\]

Match the coefficients on both sides.

\[
2a_2 = 0
\]
\[
2a_0 + 6a_3 = 0
\]
\[
(k + 4)(k + 3)a_{k+4} + (k + 3)a_{k+1} + a_k = 0
\]

Solve for \(a_2\), \(a_3\), and \(a_{k+4}\).

\[
a_2 = 0
\]
\[
a_3 = \frac{-a_0}{3}
\]
\[
a_{k+4} = -\frac{(k + 3)a_{k+1} + a_k}{(k + 4)(k + 3)}
\]

Plug in enough values of \(k\) to get four terms involving \(a_0\) and four terms involving \(a_1\).

\[
k = 0: \quad a_4 = \frac{3a_1 + a_0}{4 \cdot 3} = \frac{-a_0}{12} - \frac{a_1}{4}
\]
\[
k = 1: \quad a_5 = \frac{-4a_2 + a_1}{5 \cdot 4} = \frac{a_1}{20}
\]
\[
k = 2: \quad a_6 = \frac{-5a_3 + a_2}{6 \cdot 5} = \frac{a_0}{18}
\]
\[
k = 3: \quad a_7 = \frac{-6a_4 + a_3}{7 \cdot 6} = \frac{a_0}{252} + \frac{a_1}{28}
\]

\[
\vdots
\]

Therefore,

\[
y(t) = \sum_{n=0}^{\infty} a_n(x - 1)^n
\]
\[
= a_0 + a_1(x - 1) + a_2(x - 1)^2 + a_3(x - 1)^3 + \cdots
\]
\[
= a_0 + a_1(x - 1) - \frac{a_0}{3}(x - 1)^3 + \left(-\frac{a_0}{12} - \frac{a_1}{4}\right)(x - 1)^4 - \frac{a_1}{20}(x - 1)^5 + \frac{a_0}{18}(x - 1)^6 + \left(\frac{a_0}{252} + \frac{a_1}{28}\right)(x - 1)^7 + \cdots
\]
\[
= a_0 \left[1 - \frac{1}{3}(x - 1)^3 - \frac{1}{12}(x - 1)^4 + \frac{1}{18}(x - 1)^6 + \cdots\right] + a_1 \left[(x - 1) - \frac{1}{4}(x - 1)^4 - \frac{1}{20}(x - 1)^5 + \frac{1}{28}(x - 1)^7 + \cdots\right].
\]