Problem 20

Show directly, using the ratio test, that the two series solutions of Airy’s equation about \(x = 0\) converge for all \(x\); see Eq. (20) of the text.

Solution

The Airy equation is

\[ y'' - xy = 0. \]

\(x = 0\) is not a zero of the coefficient of \(y''\), so \(x = 0\) is an ordinary point. As such, the solution for \(y\) can be represented as a power series centered at \(x = 0\).

\[ y(x) = \sum_{n=0}^{\infty} a_n x^n \]

Differentiate this series twice with respect to \(x\) to get \(y'\) and \(y''\).

\[ y = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=1}^{\infty} na_n x^{n-1} \quad \rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \]

Substitute these series into the ODE.

\[ \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n = 0 \]

Bring \(x\) into the respective summand.

\[ \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0 \]

Substitute \(k + 1 = n - 2\) in the first sum and \(k = n\) in the second sum.

\[ \sum_{k+3=2}^{\infty} (k+3)(k+2)a_{k+3}x^{k+1} - \sum_{k=0}^{\infty} a_k x^{k+1} = 0 \]

Solve for \(k\).

\[ \sum_{k=-1}^{\infty} (k+3)(k+2)a_{k+3}x^{k+1} - \sum_{k=0}^{\infty} a_k x^{k+1} = 0 \]

Write out the first term of the first sum.

\[ 2a_2 + \sum_{k=0}^{\infty} (k+3)(k+2)a_{k+3}x^{k+1} - \sum_{k=0}^{\infty} a_k x^{k+1} = 0 \]

Now that each of the sums has the same limits and factors of \(x\), they can be combined.

\[ 2a_2 + \sum_{k=0}^{\infty} [(k+3)(k+2)a_{k+3}x^{k+1} - a_k x^{k+1}] = 0 \]

Factor the summand.

\[ 2a_2 + \sum_{k=0}^{\infty} [(k+3)(k+2)a_{k+3} - a_k]x^{k+1} = 0 + 0x + 0x^2 + \cdots \]
Match the coefficients.

\[
2a_2 = 0 \\
(k + 3)(k + 2)a_{k+3} - a_k = 0
\]

Solve for \(a_2\) and \(a_{k+3}\).

\[
a_2 = 0 \\
a_{k+3} = \frac{a_k}{(k + 3)(k + 2)}
\]

Plug in enough values of \(k\) to find a pattern.

\[
\begin{align*}
  k = 0 : & \quad a_3 = \frac{a_0}{(3)(2)} = \frac{a_0}{3 \cdot 2} \\  k = 1 : & \quad a_4 = \frac{a_1}{(4)(3)} = \frac{a_1}{4 \cdot 3} \\  k = 2 : & \quad a_5 = \frac{a_2}{(5)(4)} = 0 \\
  k = 3 : & \quad a_6 = \frac{a_3}{(6)(5)} = \frac{a_3}{6 \cdot 5 \cdot 3 \cdot 2} \\  k = 4 : & \quad a_7 = \frac{a_4}{(7)(6)} = \frac{a_4}{7 \cdot 6 \cdot 4 \cdot 3} \\  k = 5 : & \quad a_8 = \frac{a_5}{(8)(7)} = 0 \\
  k = 6 : & \quad a_9 = \frac{a_6}{(9)(8)} = \frac{a_6}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2} \\  k = 7 : & \quad a_{10} = \frac{a_7}{(10)(9)} = \frac{a_7}{10 \cdot 9 \cdot 7 \cdot 6 \cdot 4 \cdot 3} \\  k = 8 : & \quad a_{11} = \frac{a_8}{(11)(10)} = 0 \\
  \vdots & \quad \vdots \\
  \vdots & \quad \vdots 
\end{align*}
\]

Generalize these results.

\[
a_{3k} = \frac{a_0}{(3k)(3k - 3)(3k - 6) \cdots (3)\left[(3k - 1)(3k - 4)(3k - 7) \cdots (2)\right]} \\
= \frac{a_0}{(3k)! \Gamma\left(\frac{k+\frac{1}{2}}{3}\right) \Gamma\left(\frac{2}{3}\right)} \\
= \frac{a_0}{3^{2k} k! \Gamma\left(k + \frac{2}{3}\right)}
\]
\[a_{3k+1} = \frac{a_1}{(3k+1)(3k-2)(3k-5) \cdots (4)[(3k)(3k-3)(3k-6) \cdots (3)]} = \frac{a_1}{3^k \Gamma(k+\frac{4}{3})} \frac{\Gamma(\frac{3}{2})}{\Gamma(k+\frac{4}{3})} [3^k k! \Gamma(k+\frac{4}{3})] = a_1 \frac{\Gamma(\frac{4}{3})}{3^2 k! \Gamma(k+\frac{4}{3})}
\]
\[a_{3k+2} = 0\]

Therefore,
\[y(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{k=0}^{\infty} a_{3k} x^{3k} + \sum_{k=0}^{\infty} a_{3k+1} x^{3k+1} + \sum_{k=0}^{\infty} a_{3k+2} x^{3k+2}
= a_0 \sum_{k=0}^{\infty} \frac{\Gamma(\frac{3}{2})}{3^2 k! \Gamma(k+\frac{4}{3})} x^{3k} + a_1 \sum_{k=0}^{\infty} \frac{\Gamma(\frac{4}{3})}{3^2 k! \Gamma(k+\frac{4}{3})} x^{3k+1}.
\]

Now apply the ratio test to show that the first series solution converges.

\[\lim_{k \to \infty} \left| \frac{A_{k+1}}{A_k} \right| = \lim_{k \to \infty} \left| \frac{\frac{\Gamma(\frac{4}{3})}{3^2 (k+1)! \Gamma(k+\frac{4}{3})} x^{3(k+1)}}{\frac{\Gamma(\frac{4}{3})}{3^2 k! \Gamma(k+\frac{4}{3})} x^{3k}} \right|
= \lim_{k \to \infty} \frac{1}{3^2} \frac{k!}{(k+1)!} \frac{k+\frac{2}{3}}{k} \frac{\Gamma(k+\frac{4}{3})}{\Gamma(k+\frac{4}{3}+1)} x^3
= \lim_{k \to \infty} \frac{1}{3^2} \frac{k!}{(k+1)!} \frac{k+\frac{2}{3}}{k} \frac{\Gamma(k+\frac{4}{3})}{\Gamma(k+\frac{4}{3}+1)} x^3
\]
\[= \lim_{k \to \infty} \frac{1}{3^2} \frac{k+1}{k+\frac{2}{3}} \frac{1}{(k+1)!} \frac{1}{x^3}
= \lim_{k \to \infty} \frac{1}{9} \frac{k+1}{k+\frac{2}{3}} \frac{1}{x^3}
= \lim_{k \to \infty} \frac{1}{9} \frac{k+1}{k+\frac{2}{3}} |x^3|
= 0|x^3|
\]

According to this test, the first series is
\[
\begin{cases}
\text{convergent} & \text{if } 0|\!\!x^3| < 1 \\
\text{unknown} & \text{if } 0|\!\!x^3| = 1 \\
\text{divergent} & \text{if } 0|\!\!x^3| > 1
\end{cases}
\]

From the condition of convergence, which can also be written as $|x^3| < 1/0 = \infty$, or $-\infty < x^3 < \infty$, or $-\infty < x < \infty$, we see that the center of convergence is at $x = 0$ and the radius

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of convergence is $\infty$. Now apply the ratio test to the second series.

$$\lim_{k \to \infty} \left| \frac{B_{k+1}}{B_k} \right| = \lim_{k \to \infty} \left| \frac{\Gamma\left(\frac{4}{3}\right) 3^{2(k+1)} (k+1)! \Gamma\left(k+\frac{4}{3}\right) x^{3(k+1)+1}}{\Gamma\left(\frac{4}{3}\right) 3^{2k} k! \Gamma\left(k+\frac{4}{3}\right) x^{3k+1}} \right|
$$

$$= \lim_{k \to \infty} \left| \frac{\Gamma\left(\frac{4}{3}\right) 3^{2k} k! \Gamma\left(k+\frac{4}{3}\right) x^{3k+1}}{\Gamma\left(\frac{4}{3}\right) 3^{2(k+1)} (k+1)! \Gamma\left(k+\frac{4}{3}\right) x^{3k+1}} \right|
$$

$$= \lim_{k \to \infty} \left| \frac{1}{k+1} \frac{1}{k+\frac{4}{3}} x^3 \right|
$$

$$= \lim_{k \to \infty} \frac{1}{9(k+1) (k+\frac{4}{3})} |x^3|
$$

$$= 0 |x^3|
$$

According to this test, the second series is

$$\begin{cases}
\text{convergent} & \text{if } 0 |x^3| < 1 \\
\text{unknown} & \text{if } 0 |x^3| = 1 \\
\text{divergent} & \text{if } 0 |x^3| > 1
\end{cases}
$$

From the condition of convergence, which can also be written as $|x^3| < 1/0 = \infty$, or $-\infty < x^3 < \infty$, or $-\infty < x < \infty$, we see that the center of convergence is at $x = 0$ and the radius of convergence is $\infty$. 

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