Problem 3

In each of Problems 1 through 4, determine $\phi''(x_0)$, $\phi'''(x_0)$, and $\phi^{(4)}(x_0)$ for the given point x_0 if $y = \phi(x)$ is a solution of the given initial value problem.

$$x^2y'' + (1+x)y' + 3(\ln x)y = 0; \qquad y(1) = 2, \quad y'(1) = 0$$

Solution

Solve the ODE for y''.

$$y'' = -\frac{(1+x)y' + 3(\ln x)y}{x^2} \tag{1}$$

Plug in $x = 1$.

$$y''(1) = -\frac{(2)y'(1) + 3(0)y(1)}{1^2} = 0$$

Differentiate both sides of equation (1) with respect to x.

$$y''' = -\frac{\left[y' + (1+x)y'' + \frac{3}{x}y + 3(\ln x)y'\right]x^2 - 2x[(1+x)y' + 3(\ln x)y]}{x^4} \tag{2}$$

Plug in $x = 1$.

$$y'''(1) = -\frac{\left[y'(1) + (2)y''(1) + \frac{3}{1}y(1) + 3(0)y'(1)\right]1^2 - 2(1)[(2)y'(1) + 3(0)y(1)]}{1^4} = -\frac{3(2)}{1} = -6$$

Simplify equation (2) before differentiating it again.

$$y''' = \frac{2x[(1+x)y' + 3(\ln x)y] - \left[y' + (1+x)y'' + \frac{3}{x}y + 3(\ln x)y'\right]x^2}{x^4}$$

$$y''' = \frac{2[(1+x)y' + 3(\ln x)y]}{x^3} - \frac{y' + (1+x)y'' + \frac{3}{x}y + 3(\ln x)y'}{x^2}$$

$$y''' = \frac{2[(1+x)y' + 3(\ln x)y]}{x^3} - \frac{(1+x)y'' + \frac{3}{x}y + (1+3\ln x)y'}{x^2}$$

Now differentiate both sides with respect to x.

$$\begin{aligned} y^{(4)} = &\frac{2\left[y' + (1+x)y'' + \frac{3}{x}y + 3(\ln x)y'\right]x^3 - 3x^2\{2[(1+x)y' + 3(\ln x)y]\}}{x^6} \\ &- \frac{\left[y'' + (1+x)y''' - \frac{3}{x^2}y + \frac{3}{x}y' + \frac{3}{x}y' + (1+3\ln x)y''\right]x^2 - 2x[(1+x)y'' + \frac{3}{x}y + (1+3\ln x)y']}{x^4} \end{aligned}$$

Plug in $x = 1$.

$$\begin{aligned} y^{(4)}(1) &= \frac{2\left[y'(1) + (2)y''(1) + \frac{3}{1}y(1) + 3(0)y'(1)\right]1^3 - 3\{2[(2)y'(1) + 3(0)y(1)]\}}{1^6} \\ &\quad - \frac{\left[y''(1) + (2)y'''(1) - \frac{3}{1}y(1) + \frac{3}{1}y'(1) + \frac{3}{1}y'(1) + (1)y''(1)\right]1^2 - 2[(2)y''(1) + \frac{3}{1}y(1) + (1)y'(1)]}{1^4} \\ &= \frac{2[3(2)]}{1} - \frac{(2)(-6) - \frac{3}{1}(2) - 2\left[\frac{3}{1}(2)\right]}{1} \\ &= 42 \end{aligned}$$