Problem 10

The Chebyshev Equation. The Chebyshev[7] differential equation is

$$(1 - x^2)y'' - xy' + \alpha^2 y = 0,$$

where α is a constant.

(a) Determine two solutions in powers of x for $|x| < 1$, and show that they form a fundamental set of solutions.

(b) Show that if α is a nonnegative integer n, then there is a polynomial solution of degree n. These polynomials, when properly normalized, are called the Chebyshev polynomials. They are very useful in problems that require a polynomial approximation to a function defined on $-1 \leq x \leq 1$.

(c) Find a polynomial solution for each of the cases $\alpha = n = 0, 1, 2, 3$.

Solution

$x = 0$ is not a zero of the coefficient of y'', so it is an ordinary point. As such, the solution for y can be represented as a power series centered at $x = 0$.

$$y(x) = \sum_{p=0}^{\infty} a_p x^p$$

Note that because $x = -1$ and $x = 1$ are zeros of $1 - x^2$, this solution will be valid for $-1 < x < 1$. Differentiate it with respect to x twice to get y' and y''.

$$y = \sum_{p=0}^{\infty} a_p x^p \quad \rightarrow \quad y' = \sum_{p=1}^{\infty} p a_p x^{p-1} \quad \rightarrow \quad y'' = \sum_{p=2}^{\infty} p(p-1) a_p x^{p-2}$$

Substitute these expressions into the ODE.

$$(1 - x^2) \sum_{p=2}^{\infty} p(p-1) a_p x^{p-2} - x \sum_{p=1}^{\infty} p a_p x^{p-1} + \alpha^2 \sum_{p=0}^{\infty} a_p x^p = 0$$

$$\sum_{p=2}^{\infty} p(p-1) a_p x^{p-2} - x^2 \sum_{p=2}^{\infty} p(p-1) a_p x^{p-2} - x \sum_{p=1}^{\infty} p a_p x^{p-1} + \alpha^2 \sum_{p=0}^{\infty} a_p x^p = 0$$

Bring x^2, x, and α^2 into the respective summands.

$$\sum_{p=2}^{\infty} p(p-1) a_p x^{p-2} - \sum_{p=2}^{\infty} p(p-1) a_p x^p - \sum_{p=1}^{\infty} p a_p x^p + \sum_{p=0}^{\infty} \alpha^2 a_p x^p = 0$$

[7]Pafnuty L. Chebyshev (1821–1894), the most influential nineteenth-century Russian mathematician, was for 35 years professor at the University of St. Petersburg, which produced a long line of distinguished mathematicians. His study of Chebyshev polynomials began in about 1854 as part of an investigation of the approximation of functions by polynomials. Chebyshev is also known for his work in number theory and probability.

Because of the p and $p-1$ factors in the summand of the second sum, the lower limit can be set to start from $p=0$. Similarly, start the third sum from $p=0$.

$$\sum_{p=2}^{\infty} p(p-1)a_p x^{p-2} - \sum_{p=0}^{\infty} p(p-1)a_p x^p - \sum_{p=0}^{\infty} p a_p x^p + \sum_{p=0}^{\infty} \alpha^2 a_p x^p = 0$$

Substitute $k = p-2$ in the first sum and substitute $k = p$ in the others.

$$\sum_{k+2=2}^{\infty} (k+2)(k+1)a_{k+2}x^k - \sum_{k=0}^{\infty} k(k-1)a_k x^k - \sum_{k=0}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} \alpha^2 a_k x^k = 0$$

$$\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k - \sum_{k=0}^{\infty} k(k-1)a_k x^k - \sum_{k=0}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} \alpha^2 a_k x^k = 0$$

Now that the limits and the factors of x are the same in each sum, they can be combined.

$$\sum_{k=0}^{\infty} [(k+2)(k+1)a_{k+2}x^k - k(k-1)a_k x^k - k a_k x^k + \alpha^2 a_k x^k] = 0$$

Factor the summand.

$$\sum_{k=0}^{\infty} [(k+2)(k+1)a_{k+2} - k(k-1)a_k - k a_k + \alpha^2 a_k] x^k = 0$$

$$\sum_{k=0}^{\infty} [(k+2)(k+1)a_{k+2} - (k^2-\alpha^2)a_k] x^k = 0$$

The coefficients must be zero.

$$(k+2)(k+1)a_{k+2} - (k^2-\alpha^2)a_k = 0$$

Solve for a_{k+2}.

$$a_{k+2} = \frac{k^2-\alpha^2}{(k+2)(k+1)} a_k$$

Plug in enough values of k to get four terms involving a_0 and four terms involving a_1.

$$k=0: \quad a_2 = \frac{0-\alpha^2}{(2)(1)}a_0 = -\frac{\alpha^2}{2\cdot 1}a_0$$

$$k=2: \quad a_4 = \frac{4-\alpha^2}{(4)(3)}a_2 = -\frac{(4-\alpha^2)\alpha^2}{4\cdot 3\cdot 2\cdot 1}a_0$$

$$k=4: \quad a_6 = \frac{16-\alpha^2}{(6)(5)}a_4 = -\frac{(16-\alpha^2)(4-\alpha^2)\alpha^2}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}a_0$$

$$\vdots$$

$$k=1: \quad a_3 = \frac{1-\alpha^2}{(3)(2)}a_1 = \frac{1-\alpha^2}{3\cdot 2}a_1$$

$$k=3: \quad a_5 = \frac{9-\alpha^2}{(5)(4)}a_3 = \frac{(9-\alpha^2)(1-\alpha^2)}{5\cdot 4\cdot 3\cdot 2}a_1$$

$$k=5: \quad a_7 = \frac{25-\alpha^2}{(7)(6)}a_5 = \frac{(25-\alpha^2)(9-\alpha^2)(1-\alpha^2)}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}a_1$$

$$\vdots$$

Generalize the coefficients.

$$a_{2k} = \frac{[(2k-2)^2-\alpha^2][(2k-4)^2-\alpha^2]\cdots[(0)^2-\alpha^2]}{(2k)!}a_0$$

$$a_{2k+1} = \frac{[(2k-1)^2-\alpha^2][(2k-3)^2-\alpha^2]\cdots[(1)^2-\alpha^2]}{(2k+1)!}a_1$$

Therefore,

$$\begin{aligned}
y(x) &= \sum_{p=0}^{\infty} a_p x^p \\
&= a_0 + a_1 x + \sum_{k=1}^{\infty} a_{2k} x^{2k} + \sum_{k=1}^{\infty} a_{2k+1} x^{2k+1} \\
&= a_0 + a_1 x + a_0 \sum_{k=1}^{\infty} \frac{[(2k-2)^2-\alpha^2][(2k-4)^2-\alpha^2]\cdots[(0)^2-\alpha^2]}{(2k)!} x^{2k} \\
&\quad + a_1 \sum_{k=1}^{\infty} \frac{[(2k-1)^2-\alpha^2][(2k-3)^2-\alpha^2]\cdots[(1)^2-\alpha^2]}{(2k+1)!} x^{2k+1} \\
&= a_0 \left\{1 + \sum_{k=1}^{\infty} \frac{[(2k-2)^2-\alpha^2][(2k-4)^2-\alpha^2]\cdots[(0)^2-\alpha^2]}{(2k)!} x^{2k}\right\} \\
&\quad + a_1 \left\{x + \sum_{k=1}^{\infty} \frac{[(2k-1)^2-\alpha^2][(2k-3)^2-\alpha^2]\cdots[(1)^2-\alpha^2]}{(2k+1)!} x^{2k+1}\right\} \\
&= a_0 \left[1 - \frac{\alpha^2}{2\cdot 1}x^2 - \frac{(4-\alpha^2)\alpha^2}{4\cdot 3\cdot 2\cdot 1}x^4 - \frac{(16-\alpha^2)(4-\alpha^2)\alpha^2}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}x^6 - \cdots\right] \\
&\quad + a_1 \left[x + \frac{1-\alpha^2}{3\cdot 2}x^3 + \frac{(9-\alpha^2)(1-\alpha^2)}{5\cdot 4\cdot 3\cdot 2}x^5 + \frac{(25-\alpha^2)(9-\alpha^2)(1-\alpha^2)}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}x^7 + \cdots\right] \\
&= a_0 y_1(x) + a_1 y_2(x).
\end{aligned}$$

Now calculate the Wronskian of y_1 and y_2.

$$\begin{aligned}
W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\
&= y_1 y_2' - y_1' y_2 \\
&= \left[1 - \frac{\alpha^2}{2\cdot 1}x^2 - \frac{(4-\alpha^2)\alpha^2}{4\cdot 3\cdot 2\cdot 1}x^4 - \frac{(16-\alpha^2)(4-\alpha^2)\alpha^2}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}x^6 - \cdots\right] \\
&\quad \times \left[1 + \frac{1-\alpha^2}{3\cdot 2}(3x^2) + \frac{(9-\alpha^2)(1-\alpha^2)}{5\cdot 4\cdot 3\cdot 2}(5x^4) + \frac{(25-\alpha^2)(9-\alpha^2)(1-\alpha^2)}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}(7x^6) + \cdots\right] \\
&\qquad - \left[-\frac{\alpha^2}{2\cdot 1}(2x) - \frac{(4-\alpha^2)\alpha^2}{4\cdot 3\cdot 2\cdot 1}(4x^3) - \frac{(16-\alpha^2)(4-\alpha^2)\alpha^2}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}(6x^5) - \cdots\right] \\
&\qquad \times \left[x + \frac{1-\alpha^2}{3\cdot 2}x^3 + \frac{(9-\alpha^2)(1-\alpha^2)}{5\cdot 4\cdot 3\cdot 2}x^5 + \frac{(25-\alpha^2)(9-\alpha^2)(1-\alpha^2)}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}x^7 + \cdots\right]
\end{aligned}$$

At $x = 0$ the Wronskian is nonzero,

$$W(y_1, y_2)(0) = (1)(1) - (0)(0) = 1,$$

which means that y_1 and y_2 form a fundamental set of solutions for the ODE. Notice that if α is a nonnegative even number $(0, 2, 4, \ldots)$, then the series solution y_1 eventually terminates. On the other hand, if α is a nonnegative odd number $(1, 3, 5, \ldots)$, then the series solution y_2 eventually terminates.

The first few polynomial solutions are

$$\begin{aligned}
\alpha = 0: &\quad y_1(x) = 1 \\
\alpha = 1: &\quad y_2(x) = x \\
\alpha = 2: &\quad y_1(x) = 1 - 2x^2 \\
\alpha = 3: &\quad y_2(x) = x - \frac{4}{3}x^3.
\end{aligned}$$