Problem 22

The Legendre Equation. Problems 22 through 29 deal with the Legendre[8] equation

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0.$$

As indicated in Example 3, the point $x = 0$ is an ordinary point of this equation, and the distance from the origin to the nearest zero of $P(x) = 1 - x^2$ is 1. Hence the radius of convergence of series solutions about $x = 0$ is at least 1. Also notice that we need to consider only $\alpha > -1$ because if $\alpha \le -1$, then the substitution $\alpha = -(1+\gamma)$, where $\gamma \ge 0$, leads to the Legendre equation $(1-x^2)y'' - 2xy' + \gamma(\gamma+1)y = 0$.

Show that two solutions of the Legendre equation for $|x| < 1$ are

$$\begin{aligned} y_1(x) &= 1 - \frac{\alpha(\alpha+1)}{2!}x^2 + \frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!}x^4 \\ &\quad + \sum_{m=3}^{\infty}(-1)^m \frac{\alpha\cdots(\alpha-2m+2)(\alpha+1)\cdots(\alpha+2m-1)}{(2m)!}x^{2m}, \\ y_2(x) &= x - \frac{(\alpha-1)(\alpha+2)}{3!}x^3 + \frac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)}{5!}x^5 \\ &\quad + \sum_{m=3}^{\infty}(-1)^m \frac{(\alpha-1)\cdots(\alpha-2m+1)(\alpha+2)\cdots(\alpha+2m)}{(2m+1)!}x^{2m+1}. \end{aligned}$$

Solution

$x = 0$ is not one of the zeros of the coefficient of y'', so $x = 0$ is an ordinary point. As such, the solution can be represented as a power series about $x = 0$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate it with respect to x twice to get y' and y''.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Plug these expressions into the ODE.

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

[8] Adrien-Marie Legendre (1752–1833) held various positions in the French Académie des Sciences from 1783 onward. His primary work was in the fields of elliptic functions and number theory. The Legendre functions, solutions of Legendre's equation, first appeared in 1784 in his study of the attraction of spheroids.

Bring x^2, $2x$, and $\alpha(\alpha+1)$ into the respective summands.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} \alpha(\alpha+1)a_n x^n = 0$$

Because of the factors of n and $n-1$ in the second summand, n can be set to start from 0 rather than 2. Similarly, the third sum can be set to start from $n=0$ because of the factor of n.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} \alpha(\alpha+1)a_n x^n = 0$$

Make the substitution $k = n-2$ in the first sum and $k = n$ in the others.

$$\sum_{k+2=2}^{\infty} (k+2)(k+1)a_{k+2} x^k - \sum_{k=0}^{\infty} k(k-1)a_k x^k - \sum_{k=0}^{\infty} 2ka_k x^k + \sum_{k=0}^{\infty} \alpha(\alpha+1)a_k x^k = 0$$

Solve for k.

$$\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k - \sum_{k=0}^{\infty} k(k-1)a_k x^k - \sum_{k=0}^{\infty} 2ka_k x^k + \sum_{k=0}^{\infty} \alpha(\alpha+1)a_k x^k = 0$$

Now that the limits are the same in each sum, they can be combined.

$$\sum_{k=0}^{\infty} \left[(k+2)(k+1)a_{k+2} x^k - k(k-1)a_k x^k - 2ka_k x^k + \alpha(\alpha+1)a_k x^k\right] = 0$$

Factor the summand.

$$\sum_{k=0}^{\infty} \left[(k+2)(k+1)a_{k+2} - k(k-1)a_k - 2ka_k + \alpha(\alpha+1)a_k\right] x^k = 0$$

$$\sum_{k=0}^{\infty} \left\{(k+2)(k+1)a_{k+2} - [k(k+1) - \alpha(\alpha+1)]a_k\right\} x^k = 0$$

The coefficients must be zero.

$$(k+2)(k+1)a_{k+2} - [k(k+1) - \alpha(\alpha+1)]a_k = 0$$

Solve for a_{k+2}.

$$a_{k+2} = \frac{k(k+1) - \alpha(\alpha+1)}{(k+2)(k+1)} a_k$$

Plug in enough values of k to see a pattern and determine a_k.

$$k=0: \quad a_2 = \frac{0(1) - \alpha(\alpha+1)}{(2)(1)} a_0 = -\frac{\alpha(\alpha+1)}{2\cdot 1} a_0$$

$$k=1: \quad a_3 = \frac{1(2) - \alpha(\alpha+1)}{(3)(2)} a_1 = -\frac{(\alpha-1)(\alpha+2)}{3\cdot 2} a_1$$

$$k=2: \quad a_4 = \frac{2(3) - \alpha(\alpha+1)}{(4)(3)} a_2 = \frac{(\alpha-2)(\alpha+3)\alpha(\alpha+1)}{4\cdot 3\cdot 2\cdot 1} a_0$$

$$k=3: \quad a_5 = \frac{3(4) - \alpha(\alpha+1)}{(5)(4)} a_3 = \frac{(\alpha-3)(\alpha+4)(\alpha-1)(\alpha+2)}{5\cdot 4\cdot 3\cdot 2} a_1$$

$$k = 4: \quad a_6 = \frac{4(5) - \alpha(\alpha+1)}{(6)(5)} a_4 = -\frac{(\alpha-4)(\alpha+5)(\alpha-2)(\alpha+3)\alpha(\alpha+1)}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} a_0$$

$$k = 5: \quad a_7 = \frac{5(6) - \alpha(\alpha+1)}{(7)(6)} a_5 = -\frac{(\alpha-5)(\alpha+6)(\alpha-3)(\alpha+4)(\alpha-1)(\alpha+2)}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} a_1$$

$$\vdots$$

$$a_{2k} = (-1)^k \frac{\alpha \cdots (\alpha - 2k + 2) \cdot (\alpha + 1) \cdots (\alpha + 2k - 1)}{(2k)!} a_0$$

$$a_{2k+1} = (-1)^k \frac{(\alpha - 1) \cdots (\alpha - 2k + 1) \cdot (\alpha + 2) \cdots (\alpha + 2k)}{(2k+1)!} a_1$$

Therefore,

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + \sum_{k=1}^{\infty} a_{2k} x^{2k} + \sum_{k=1}^{\infty} a_{2k+1} x^{2k+1} \\
&= a_0 + a_1 x + \sum_{k=1}^{\infty} (-1)^k \frac{\alpha \cdots (\alpha - 2k + 2) \cdot (\alpha + 1) \cdots (\alpha + 2k - 1)}{(2k)!} a_0 x^{2k} \\
&\qquad + \sum_{k=1}^{\infty} (-1)^k \frac{(\alpha - 1) \cdots (\alpha - 2k + 1) \cdot (\alpha + 2) \cdots (\alpha + 2k)}{(2k+1)!} a_1 x^{2k+1} \\
&= a_0 \left[1 + \sum_{k=1}^{\infty} (-1)^k \frac{\alpha \cdots (\alpha - 2k + 2) \cdot (\alpha + 1) \cdots (\alpha + 2k - 1)}{(2k)!} x^{2k} \right] \\
&\qquad + a_1 \left[x + \sum_{k=1}^{\infty} (-1)^k \frac{(\alpha - 1) \cdots (\alpha - 2k + 1) \cdot (\alpha + 2) \cdots (\alpha + 2k)}{(2k+1)!} x^{2k+1} \right] \\
&= a_0 y_1(x) + a_1 y_2(x).
\end{aligned}$$