Problem 23

The Legendre Equation. Problems 22 through 29 deal with the Legendre[8] equation

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0.$$

As indicated in Example 3, the point $x = 0$ is an ordinary point of this equation, and the distance from the origin to the nearest zero of $P(x) = 1 - x^2$ is 1. Hence the radius of convergence of series solutions about $x = 0$ is at least 1. Also notice that we need to consider only $\alpha > -1$ because if $\alpha \leq -1$, then the substitution $\alpha = -(1+\gamma)$, where $\gamma \geq 0$, leads to the Legendre equation $(1-x^2)y'' - 2xy' + \gamma(\gamma+1)y = 0$.

Show that if α is zero or a positive even integer $2n$, the series solution y_1 reduces to a polynomial of degree $2n$ containing only even powers of x. Find the polynomials corresponding to $\alpha = 0, 2,$ and 4. Show that if α is a positive odd integer $2n+1$, the series solution y_2 reduces to a polynomial of degree $2n+1$ containing only odd powers of x. Find the polynomials corresponding to $\alpha = 1, 3,$ and 5.

Solution

The general solution to the Legendre equation was found in Problem 22 to be

$$\begin{aligned} y(x) &= a_0\left[1 + \sum_{k=1}^{\infty}(-1)^k \frac{\alpha\cdots(\alpha-2k+2)\cdot(\alpha+1)\cdots(\alpha+2k-1)}{(2k)!}x^{2k}\right] \\ &\qquad + a_1\left[x + \sum_{k=1}^{\infty}(-1)^k \frac{(\alpha-1)\cdots(\alpha-2k+1)\cdot(\alpha+2)\cdots(\alpha+2k)}{(2k+1)!}x^{2k+1}\right] \\ &= a_0 y_1(x) + a_1 y_2(x). \end{aligned}$$

Suppose that α is an even integer $2n$, where $n = 0, 1, 2, \ldots$.

$$\begin{aligned} y(x) &= a_0\left[1 + \sum_{k=1}^{\infty}(-1)^k \frac{2n\cdots(2n-2k+2)\cdot(2n+1)\cdots(2n+2k-1)}{(2k)!}x^{2k}\right] \\ &\qquad + a_1\left[x + \sum_{k=1}^{\infty}(-1)^k \frac{(2n-1)\cdots(2n-2k+1)\cdot(2n+2)\cdots(2n+2k)}{(2k+1)!}x^{2k+1}\right] \end{aligned}$$

Observe that if $2n - 2k + 2 = 0$, or $k = n + 1$, then the series solution for y_1 terminates. The last term in the series obtained then is $k = n$.

$$\begin{aligned} y(x) &= a_0\left[1 + \cdots + (-1)^n \frac{(2n)\cdots(2)\cdot(2n+1)\cdots(4n-1)}{(2n)!}x^{2n}\right] \\ &\qquad + a_1\left[x + \sum_{k=1}^{\infty}(-1)^k \frac{(2n-1)\cdots(2n-2k+1)\cdot(2n+2)\cdots(2n+2k)}{(2k+1)!}x^{2k+1}\right] \end{aligned}$$

[8] Adrien-Marie Legendre (1752–1833) held various positions in the French Académie des Sciences from 1783 onward. His primary work was in the fields of elliptic functions and number theory. The Legendre functions, solutions of Legendre's equation, first appeared in 1784 in his study of the attraction of spheroids.

Below are the first few polynomial solutions for $\alpha = 0, 2, 4$ ($n = 0, 1, 2$, respectively).

$$\begin{aligned}
n = 0: &\quad y_1(x) = 1 \\
n = 1: &\quad y_1(x) = 1 + (-1)\frac{(2)(3)}{2!}x^2 = 1 - 3x^2 \\
n = 2: &\quad y_1(x) = 1 + (-1)\frac{(4)(5)}{2!}x^2 + (-1)^2\frac{(4)(2)(5)(7)}{4!}x^4 = 1 - 10x^2 + \frac{35}{3}x^4
\end{aligned}$$

Suppose instead that α is an odd integer $2n+1$, where $n = 0, 1, 2, \ldots$.

$$\begin{aligned}
y(x) = a_0 &\left[1 + \sum_{k=1}^{\infty}(-1)^k \frac{(2n+1)\cdots(2n-2k+3)\cdot(2n+2)\cdots(2n+2k)}{(2k)!}x^{2k}\right] \\
&+ a_1\left[x + \sum_{k=1}^{\infty}(-1)^k \frac{(2n)\cdots(2n-2k+2)\cdot(2n+3)\cdots(2n+2k+1)}{(2k+1)!}x^{2k+1}\right]
\end{aligned}$$

Observe that if $2n - 2k + 2 = 0$, or $k = n+1$, then the series solution for y_2 terminates. The last term in the series obtained then is $k = n$.

$$\begin{aligned}
y(x) = a_0 &\left[1 + \sum_{k=1}^{\infty}(-1)^k \frac{(2n+1)\cdots(2n-2k+3)\cdot(2n+2)\cdots(2n+2k)}{(2k)!}x^{2k}\right] \\
&+ a_1\left[x + \cdots + (-1)^n \frac{(2n)\cdots(2)\cdot(2n+3)\cdots(4n+1)}{(2n+1)!}x^{2n+1}\right]
\end{aligned}$$

Below are the first few polynomial solutions for $\alpha = 1, 3, 5$ ($n = 0, 1, 2$, respectively).

$$\begin{aligned}
n = 0: &\quad y_2(x) = x \\
n = 1: &\quad y_2(x) = x + (-1)\frac{(2)(5)}{3!}x^3 = x - \frac{5}{3}x^3 \\
n = 2: &\quad y_2(x) = x + (-1)\frac{(4)(7)}{3!}x^3 + (-1)^2\frac{(4)(2)(7)(9)}{5!}x^5 = x - \frac{14}{3}x^3 + \frac{21}{5}x^5
\end{aligned}$$