Problem 27

The Legendre Equation. Problems 22 through 29 deal with the Legendre[8] equation

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0.$$

As indicated in Example 3, the point $x = 0$ is an ordinary point of this equation, and the distance from the origin to the nearest zero of $P(x) = 1 - x^2$ is 1. Hence the radius of convergence of series solutions about $x = 0$ is at least 1. Also notice that we need to consider only $\alpha > -1$ because if $\alpha \leq -1$, then the substitution $\alpha = -(1+\gamma)$, where $\gamma \geq 0$, leads to the Legendre equation $(1-x^2)y'' - 2xy' + \gamma(\gamma+1)y = 0$.

Show that for $n = 0, 1, 2, 3$, the corresponding Legendre polynomial is given by

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

This formula, known as Rodrigues's formula,[9] is true for all positive integers n.

Solution

$$n = 0: \quad P_0(x) = \frac{1}{2^0 0!}\frac{d^0}{dx^0}(x^2-1)^0 = \frac{1}{2^0 0!}(x^2-1)^0 = \frac{1}{1}(1) = 1$$

$$n = 1: \quad P_1(x) = \frac{1}{2^1 1!}\frac{d^1}{dx^1}(x^2-1)^1 = \frac{1}{2}\frac{d}{dx}(x^2-1) = \frac{1}{2}(2x) = x$$

$$n = 2: \quad P_2(x) = \frac{1}{2^2 2!}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\frac{d}{dx}[2(x^2-1)\cdot 2x] = \frac{1}{8}\frac{d}{dx}(4x^3-4x) = \frac{1}{2}(3x^2-1)$$

$$\begin{aligned} n = 3: \quad P_3(x) = \frac{1}{2^3 3!}\frac{d^3}{dx^3}(x^2-1)^3 &= \frac{1}{48}\frac{d^2}{dx^2}[3(x^2-1)^2\cdot 2x] \\ &= \frac{1}{48}\frac{d}{dx}[6(x^2-1)\cdot(2x)^2 + 3(x^2-1)^2\cdot 2] \\ &= \frac{1}{48}\frac{d}{dx}[24x^4 - 24x^2 + 6(x^2-1)^2] \\ &= \frac{1}{48}[96x^3 - 48x + 12(x^2-1)\cdot 2x] \\ &= 2x^3 - x + \frac{x}{2}(x^2-1) \\ &= 2x^3 - x + \frac{x^3}{2} - \frac{x}{2} \\ &= \frac{5}{2}x^3 - \frac{3x}{2} \\ &= -\frac{3}{2}\left(x - \frac{5}{3}x^3\right) \end{aligned}$$

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[8]Adrien-Marie Legendre (1752–1833) held various positions in the French Académie des Sciences from 1783 onward. His primary work was in the fields of elliptic functions and number theory. The Legendre functions, solutions of Legendre's equation, first appeared in 1784 in his study of the attraction of spheroids.

[9]Benjamin Olinde Rodrigues (1795–1851) published this result as part of his doctoral thesis from the University of Paris in 1815. He then became a banker and social reformer but retained an interest in mathematics. Unfortunately, his later papers were not appreciated until the late twentieth century.