Problem 29

The Legendre Equation. Problems 22 through 29 deal with the Legendre[8] equation

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0.$$

As indicated in Example 3, the point $x = 0$ is an ordinary point of this equation, and the distance from the origin to the nearest zero of $P(x) = 1 - x^2$ is 1. Hence the radius of convergence of series solutions about $x = 0$ is at least 1. Also notice that we need to consider only $\alpha > -1$ because if $\alpha \leq -1$, then the substitution $\alpha = -(1+\gamma)$, where $\gamma \geq 0$, leads to the Legendre equation $(1-x^2)y'' - 2xy' + \gamma(\gamma+1)y = 0$.

Given a polynomial f of degree n, it is possible to express f as a linear combination of P_0, P_1, P_2, $\ldots$, P_n:

$$f(x) = \sum_{k=0}^{n} a_k P_k(x).$$

Using the result of Problem 28, show that

$$a_k = \frac{2k+1}{2} \int_{-1}^{1} f(x) P_k(x)\, dx.$$

Solution

Consider the expansion of $f(x)$ in terms of Legendre polynomials.

$$f(x) = \sum_{k=0}^{n} a_k P_k(x)$$

Multiply both sides by $P_m(x)$, where m is an integer such that $0 \leq m \leq n$.

$$f(x)P_m(x) = \sum_{k=0}^{n} a_k P_k(x) P_m(x)$$

Integrate both sides with respect to x from -1 to 1.

$$\int_{-1}^{1} f(x)P_m(x)\, dx = \int_{-1}^{1} \sum_{k=0}^{n} a_k P_k(x) P_m(x)\, dx$$

The integral of a sum is the sum of the integrals.

$$\int_{-1}^{1} f(x)P_m(x)\, dx = \sum_{k=0}^{n} \int_{-1}^{1} a_k P_k(x) P_m(x)\, dx$$

Bring the constants in front.

$$\int_{-1}^{1} f(x)P_m(x)\, dx = \sum_{k=0}^{n} a_k \int_{-1}^{1} P_k(x) P_m(x)\, dx$$

[8] Adrien-Marie Legendre (1752–1833) held various positions in the French Académie des Sciences from 1783 onward. His primary work was in the fields of elliptic functions and number theory. The Legendre functions, solutions of Legendre's equation, first appeared in 1784 in his study of the attraction of spheroids.

Because the Legendre polynomials are orthogonal, the integral on the right side is zero if $k \neq m$. As a result of integrating both sides, every term in the series vanishes except for the one that corresponds to the $k = m$ term.

$$\int_{-1}^{1} f(x)P_m(x)\,dx = a_m \int_{-1}^{1} P_m(x)P_m(x)\,dx$$

Solve for a_m.

$$a_m = \frac{1}{\displaystyle\int_{-1}^{1} P_m^2(x)\,dx} \int_{-1}^{1} f(x)P_m(x)\,dx.$$

Therefore, using the fact that the integral in the denominator is $2/(2m+1)$,

$$a_m = \frac{2m+1}{2} \int_{-1}^{1} f(x)P_m(x)\,dx.$$