Problem 2

In each of Problems 1 through 4, determine \( \phi''(x_0) \), \( \phi'''(x_0) \), and \( \phi^{(4)}(x_0) \) for the given point \( x_0 \) if \( y = \phi(x) \) is a solution of the given initial value problem.

\[
y'' + (\sin x)y' + (\cos x)y = 0; \quad y(0) = 0, \quad y'(0) = 1
\]

Solution

Solve the ODE for \( y'' \).

\[
y'' = -(\sin x)y' - (\cos x)y \tag{1}
\]

Plug in \( x = 0 \).

\[
y''(0) = -(\sin 0)y'(0) - (\cos 0)y(0) = 0
\]

Differentiate both sides of equation (1) with respect to \( x \).

\[
y''' = -(\cos x)y' - (\sin x)y'' + (\sin x)y' - (\cos x)y' \tag{2}
\]

Plug in \( x = 0 \).

\[
y'''(0) = -(\cos 0)y'(0) - (\sin 0)y''(0) + (\sin 0)y(0) - (\cos 0)y'(0) = -2y'(0) = -2
\]

Differentiate both sides of equation (2) with respect to \( x \).

\[
y^{(4)} = (\sin x)y' - (\cos x)y'' - (\sin x)y'' - (\sin x)y''' + (\cos x)y + (\sin x)y' + (\sin x)y' - (\cos x)y''
\]

Plug in \( x = 0 \).

\[
y^{(4)}(0) = (\sin 0)y'(0) - (\cos 0)y''(0) - (\cos 0)y''(0) - (\sin 0)y'''(0) + (\cos 0)y(0)
\]
\[
+ (\sin 0)y'(0) + (\sin 0)y'(0) - (\cos 0)y''(0) = -3y''(0) + y(0) = 0
\]