Problem 22

The Legendre Equation. Problems 22 through 29 deal with the Legendre\(^8\) equation

\[
(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0.
\]

As indicated in Example 3, the point \(x = 0\) is an ordinary point of this equation, and the distance from the origin to the nearest zero of \(P(x) = 1 - x^2\) is 1. Hence the radius of convergence of series solutions about \(x = 0\) is at least 1. Also notice that we need to consider only \(\alpha > -1\) because if \(\alpha \leq -1\), then the substitution \(\alpha = -(1 + \gamma)\), where \(\gamma \geq 0\), leads to the Legendre equation

\[
(1 - x^2)y'' - 2xy' + \gamma(\gamma + 1)y = 0.
\]

Show that two solutions of the Legendre equation for \(|x| < 1\) are

\[
y_1(x) = 1 - \frac{\alpha(\alpha + 1)}{2!}x^2 + \frac{\alpha(\alpha - 2)(\alpha + 1)(\alpha + 3)}{4!}x^4
\]

\[+ \sum_{m=3}^{\infty} (-1)^m \alpha \cdots (\alpha - 2m + 2)(\alpha + 1) \cdots (\alpha + 2m - 1) \frac{x^{2m}}{(2m)!},\]

\[
y_2(x) = x - \frac{(\alpha - 1)(\alpha + 2)}{3!}x^3 + \frac{(\alpha - 1)(\alpha - 3)(\alpha + 2)(\alpha + 4)}{5!}x^5
\]

\[+ \sum_{m=3}^{\infty} (-1)^m (\alpha - 1) \cdots (\alpha - 2m + 1)(\alpha + 2) \cdots (\alpha + 2m) \frac{x^{2m+1}}{(2m + 1)!}.\]

\(^8\)Adrien-Marie Legendre (1752–1833) held various positions in the French Académie des Sciences from 1783 onward. His primary work was in the fields of elliptic functions and number theory. The Legendre functions, solutions of Legendre’s equation, first appeared in 1784 in his study of the attraction of spheroids.

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