Problem 3

In each of Problems 1 through 4, determine \( \phi''(x_0) \), \( \phi'''(x_0) \), and \( \phi^{(4)}(x_0) \) for the given point \( x_0 \) if \( y = \phi(x) \) is a solution of the given initial value problem.

\[
x^2y'' + (1 + x)y' + 3(\ln x)y = 0; \quad y(1) = 2, \quad y'(1) = 0
\]

Solution

Solve the ODE for \( y'' \).

\[
y'' = -\frac{(1 + x)y' + 3(\ln x)y}{x^2} \quad (1)
\]

Plug in \( x = 1 \).

\[
y''(1) = -\frac{(2)y'(1) + 3(0)y(1)}{1^2} = 0
\]

Differentiate both sides of equation (1) with respect to \( x \).

\[
y''' = -\frac{\left[y' + (1 + x)y'' + \frac{3}{x}y + 3(\ln x)y\right] x^2 - 2x[(1 + x)y' + 3(\ln x)y]}{x^4} \quad (2)
\]

Plug in \( x = 1 \).

\[
y'''(1) = -\frac{\left[y'(1) + (2)y''(1) + \frac{3}{x}y(1) + 3(0)y'(1)\right] 1^2 - 2(1)[(2)y'(1) + 3(0)y(1)]}{1^4} = \frac{-3(2)}{1} = -6
\]

Simplify equation (2) before differentiating it again.

\[
y''' = \frac{2x[(1 + x)y' + 3(\ln x)y] - \left[y' + (1 + x)y'' + \frac{3}{x}y + 3(\ln x)y\right] x^2}{x^4}
\]

\[
y'''' = \frac{2[(1 + x)y' + 3(\ln x)y]}{x^3} \quad \frac{y' + (1 + x)y'' + \frac{3}{x}y + 3(\ln x)y}{x^2}
\]

Now differentiate both sides with respect to \( x \).

\[
y^{(4)} = \frac{2\left[y' + (1 + x)y'' + \frac{3}{x}y + 3(\ln x)y\right]}{x^4} x^3 - 3x^2 \left\{2[(1 + x)y' + 3(\ln x)y]\right\}
\]

Plug in \( x = 1 \).

\[
y^{(4)}(1) = \frac{2\left[y'(1) + (2)y''(1) + \frac{3}{x}y(1) + 3(0)y'(1)\right] 1^3 - 3\left\{2[(2)y'(1) + 3(0)y(1)]\right\}}{1^6}
\]

\[
= \frac{2(3)(2)}{1} - \frac{(2)(-6) - 3(2) - 2 \left[\frac{3}{x}(2)\right]}{1} = 42
\]

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