Problem 7

In each of Problems 5 through 8, determine a lower bound for the radius of convergence of series solutions about each given point $x_0$ for the given differential equation.

$$(1 + x^3)y'' + 4xy' + y = 0; \quad x_0 = 0, \quad x_0 = 2$$

Solution

The coefficient of $y''$ is $1 + x^3$. Determine where its zeros are located.

$$1 + x^3 = 0$$

$$x^3 = -1$$

$$x = (-1)^{1/3}$$

$$= (e^{i\pi + 2in\pi})^{1/3}, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$= e^{i\pi/3 + 2n\pi/3}$$

Distinct zeros occur if $n = 0$, $n = 1$, and $n = 2$. Other values of $n$ lead to redundant values.

$$n = 0 : \quad x = e^{i\pi/3} = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$n = 1 : \quad x = e^{i\pi} = \cos \pi + i \sin \pi = -1$$

$$n = 2 : \quad x = e^{5i\pi/3} = \cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Plot these zeros in the complex plane and expand a circle centered at $x_0$ as much as possible until it intersects one of them.
To determine the radii convergence, use the distance formula for the points, \((0,0)\) and \((1/2, \sqrt{3}/2)\), in the first case and the points, \((2,0)\) and \((1/2, \sqrt{3}/2)\), in the second case.

\[
x_0 = 0 : \quad d = \sqrt{\left(\frac{1}{2} - 0\right)^2 + \left(\frac{\sqrt{3}}{2} - 0\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1
\]

\[
x_0 = 2 : \quad d = \sqrt{\left(\frac{1}{2} - 2\right)^2 + \left(\frac{\sqrt{3}}{2} - 0\right)^2} = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{3}
\]

If \(x_0 = 0\), the lower bound for the radius of convergence is 1. If \(x_0 = 2\), the lower bound for the radius of convergence is \(\sqrt{3}\).