Problem 2

In each of Problems 1 through 12, determine the general solution of the given differential equation that is valid in any interval not including the singular point.

$$(x+1)^2y'' + 3(x+1)y' + 0.75y = 0$$

Solution

This is a homogeneous equidimensional (Euler) ODE, so the solution is of the form $y = (x+1)^r$.

$$y = (x+1)^r \quad \rightarrow \quad y' = r(x+1)^{r-1} \quad \rightarrow \quad y'' = r(r-1)(x+1)^{r-2}$$

Substitute these expressions into the ODE.

$$(x+1)^2r(r-1)(x+1)^{r-2} + 3(x+1)r(x+1)^{r-1} + 0.75(x+1)^r = 0$$

$$r(r-1)(x+1)^r + 3r(x+1)^r + 0.75(x+1)^r = 0$$

Divide both sides by $(x+1)^r$.

$$r(r-1) + 3r + 0.75 = 0$$

Solve for r.

$$r^2 + 2r + 0.75 = 0$$

$$(r+1.5)(r+0.5) = 0$$

$$r = \{-1.5, -0.5\}$$

Two solutions to the ODE are $y = (x+1)^{-1.5}$ and $y = (x+1)^{-0.5}$. According to the principle of superposition, the general solution is a linear combination of these two. Therefore,

$$y(x) = C_1(x+1)^{-1.5} + C_2(x+1)^{-0.5}.$$