Problem 5

In each of Problems 1 through 12, determine the general solution of the given differential equation that is valid in any interval not including the singular point.

$$x^2y'' - xy' + y = 0$$

Solution

This is a homogeneous equidimensional (Euler) ODE, so the solution is of the form $y = x^r$.

$$y = x^r \quad \rightarrow \quad y' = rx^{r-1} \quad \rightarrow \quad y'' = r(r-1)x^{r-2}$$

Substitute these expressions into the ODE.

$$x^2r(r-1)x^{r-2} - xrx^{r-1} + x^r = 0$$

$$r(r-1)x^r - rx^r + x^r = 0$$

Divide both sides by x^r.

$$r(r-1) - r + 1 = 0$$

Solve for r.

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0$$

$$r = \{1\}$$

One solution to the ODE is $y = x^1 = x$. The general solution can be obtained by using the method of reduction of order: Plug in $y(x) = c(x)x$ into the ODE.

$$x^2[c(x)x]'' - x[c(x)x]' + [c(x)x] = 0$$

Evaluate the derivatives.

$$x^2[c'(x)x + c(x)]' - x[c'(x)x + c(x)] + [c(x)x] = 0$$

$$x^2[c''(x)x + 2c'(x)] - x[c'(x)x + c(x)] + [c(x)x] = 0$$

$$c''(x)x^3 + 2c'(x)x^2 - c'(x)x^2 - \cancel{c(x)x} + \cancel{c(x)x} = 0$$

$$c''(x)x^3 + c'(x)x^2 = 0$$

Solve this ODE for $c(x)$.

$$\frac{c''(x)}{c'(x)} = -\frac{1}{x}$$

$$\frac{d}{dx}\ln c'(x) = -\frac{1}{x}$$

Integrate both sides with respect to x.

$$\ln c'(x) = -\ln x + C_1$$

Exponentiate both sides.

$$\begin{aligned}
c'(x) &= e^{-\ln x + C_1} \\
&= e^{\ln x^{-1} + C_1} \\
&= e^{\ln x^{-1}} e^{C_1} \\
&= x^{-1} e^{C_1} \\
&= \frac{e^{C_1}}{x}
\end{aligned}$$

Integrate both sides with respect to x once more.

$$c(x) = e^{C_1} \ln x + C_2$$

Therefore, using a new constant C_3 for e^{C_1},

$$\begin{aligned}
y(x) &= c(x)x \\
&= (C_3 \ln x + C_2)x \\
&= C_3 x \ln x + C_2 x.
\end{aligned}$$