Problem 9

In each of Problems 1 through 12, determine the general solution of the given differential equation that is valid in any interval not including the singular point.

$$x^2y'' - 5xy' + 9y = 0$$

Solution

This is a homogeneous equidimensional (Euler) ODE, so the solution is of the form $y = x^r$.

$$y = x^r \quad \rightarrow \quad y' = rx^{r-1} \quad \rightarrow \quad y'' = r(r-1)x^{r-2}$$

Substitute these expressions into the ODE.

$$x^2r(r-1)x^{r-2} - 5xrx^{r-1} + 9x^r = 0$$

$$r(r-1)x^r - 5rx^r + 9x^r = 0$$

Divide both sides by x^r.

$$r(r-1) - 5r + 9 = 0$$

Solve for r.

$$r^2 - 6r + 9 = 0$$

$$(r-3)^2 = 0$$

One solution to the ODE is $y = x^3$. The general solution can be obtained by using the method of reduction of order: Plug in $y(x) = c(x)x^3$ into the ODE.

$$x^2[c(x)x^3]'' - 5x[c(x)x^3]' + 9[c(x)x^3] = 0$$

Evaluate the derivatives.

$$x^2[c'(x)x^3 + 3c(x)x^2]' - 5x[c'(x)x^3 + 3c(x)x^2] + 9[c(x)x^3] = 0$$

$$x^2[c''(x)x^3 + 6c'(x)x^2 + 6c(x)x] - 5x[c'(x)x^3 + 3c(x)x^2] + 9[c(x)x^3] = 0$$

$$c''(x)x^5 + 6c'(x)x^4 + \cancel{6c(x)x^3} - 5c'(x)x^4 - \cancel{15c(x)x^3} + \cancel{9c(x)x^3} = 0$$

$$c''(x)x^5 + c'(x)x^4 = 0$$

Solve this ODE for $c(x)$.

$$\frac{c''(x)}{c'(x)} = -\frac{1}{x}$$

$$\frac{d}{dx}\ln c'(x) = -\frac{1}{x}$$

Integrate both sides with respect to x.

$$\ln c'(x) = -\ln x + C_1$$

Exponentiate both sides.

$$\begin{aligned}
c'(x) &= e^{-\ln x + C_1} \\
&= e^{\ln x^{-1} + C_1} \\
&= e^{\ln x^{-1}} e^{C_1} \\
&= x^{-1} e^{C_1} \\
&= \frac{e^{C_1}}{x}
\end{aligned}$$

Integrate both sides with respect to x once more.

$$c(x) = e^{C_1} \ln x + C_2$$

Therefore, using a new constant C_3 for e^{C_1},

$$\begin{aligned}
y(x) &= c(x)x^3 \\
&= (C_3 \ln x + C_2)x^3 \\
&= C_3 x^3 \ln x + C_2 x^3.
\end{aligned}$$