

## Problem 11

In each of Problems 1 through 12, determine the general solution of the given differential equation that is valid in any interval not including the singular point.

$$x^2y'' + 2xy' + 4y = 0$$

### Solution

This is a homogeneous equidimensional (Euler) ODE, so the solution is of the form  $y = x^r$ .

$$y = x^r \quad \rightarrow \quad y' = rx^{r-1} \quad \rightarrow \quad y'' = r(r-1)x^{r-2}$$

Substitute these expressions into the ODE.

$$x^2r(r-1)x^{r-2} + 2xrx^{r-1} + 4x^r = 0$$

$$r(r-1)x^r + 2rx^r + 4x^r = 0$$

Divide both sides by  $x^r$ .

$$r(r-1) + 2r + 4 = 0$$

Solve for  $r$ .

$$r^2 + r + 4 = 0$$

$$r = \frac{-1 \pm \sqrt{1 - 4(1)(4)}}{2(1)} = \frac{-1 \pm \sqrt{-15}}{2} = \frac{-1 \pm i\sqrt{15}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{15}}{2}$$

$$r = \left\{ -\frac{1}{2} - i\frac{\sqrt{15}}{2}, -\frac{1}{2} + i\frac{\sqrt{15}}{2} \right\}$$

Two solutions to the ODE are  $y = x^{-1/2-i\sqrt{15}/2}$  and  $y = x^{-1/2+i\sqrt{15}/2}$ . According to the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned} y(x) &= C_1x^{-1/2-i\sqrt{15}/2} + C_2x^{-1/2+i\sqrt{15}/2} \\ &= C_1x^{-1/2}x^{-i\sqrt{15}/2} + C_2x^{-1/2}x^{i\sqrt{15}/2} \\ &= C_1x^{-1/2}e^{\ln x^{-i\sqrt{15}/2}} + C_2x^{-1/2}e^{\ln x^{i\sqrt{15}/2}} \\ &= C_1x^{-1/2}e^{(-i\sqrt{15}/2)\ln x} + C_2x^{-1/2}e^{(i\sqrt{15}/2)\ln x} \\ &= C_1x^{-1/2} \left[ \cos\left(-\frac{\sqrt{15}}{2}\ln x\right) + i\sin\left(-\frac{\sqrt{15}}{2}\ln x\right) \right] + C_2x^{-1/2} \left[ \cos\left(\frac{\sqrt{15}}{2}\ln x\right) + i\sin\left(\frac{\sqrt{15}}{2}\ln x\right) \right] \\ &= C_1x^{-1/2} \left[ \cos\left(\frac{\sqrt{15}}{2}\ln x\right) - i\sin\left(\frac{\sqrt{15}}{2}\ln x\right) \right] + C_2x^{-1/2} \left[ \cos\left(\frac{\sqrt{15}}{2}\ln x\right) + i\sin\left(\frac{\sqrt{15}}{2}\ln x\right) \right] \\ &= (C_1 + C_2)x^{-1/2} \cos\left(\frac{\sqrt{15}}{2}\ln x\right) + (-iC_1 + iC_2)x^{-1/2} \sin\left(\frac{\sqrt{15}}{2}\ln x\right) \\ &= C_3x^{-1/2} \cos\left(\frac{\sqrt{15}}{2}\ln x\right) + C_4x^{-1/2} \sin\left(\frac{\sqrt{15}}{2}\ln x\right) \end{aligned}$$