Problem 15

In each of Problems 13 through 16, find the solution of the given initial value problem. Plot the graph of the solution and describe how the solution behaves as $x \to 0$.

$$x^2y'' - 3xy' + 4y = 0, \qquad y(-1) = 2, \quad y'(-1) = 3$$

Solution

This is a homogeneous equidimensional (Euler) ODE, so the solution is of the form $y = x^r$.

$$y = x^r \quad \to \quad y' = rx^{r-1} \quad \to \quad y'' = r(r-1)x^{r-2}$$

Substitute these expressions into the ODE.

$$x^2r(r-1)x^{r-2} - 3xrx^{r-1} + 4x^r = 0$$

$$r(r-1)x^r - 3rx^r + 4x^r = 0$$

Divide both sides by x^r.

$$r(r-1) - 3r + 4 = 0$$

Solve for r.

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0$$

$$r = \{2\}$$

One solution to the ODE is $y = x^2$. The general solution can be obtained by using the method of reduction of order: Plug in $y(x) = c(x)x^2$ into the ODE.

$$x^2[c(x)x^2]'' - 3x[c(x)x^2]' + 4[c(x)x^2] = 0$$

Evaluate the derivatives.

$$x^2[c'(x)x^2 + 2c(x)x]' - 3x[c'(x)x^2 + 2c(x)x] + 4[c(x)x^2] = 0$$

$$x^2[c''(x)x^2 + 4c'(x)x + 2c(x)] - 3x[c'(x)x^2 + 2c(x)x] + 4[c(x)x^2] = 0$$

$$c''(x)x^4 + 4c'(x)x^3 + \cancel{2c(x)x^2} - 3c'(x)x^3 - \cancel{6c(x)x^2} + \cancel{4c(x)x^2} = 0$$

$$c''(x)x^4 + c'(x)x^3 = 0$$

Solve this ODE for $c(x)$.

$$\frac{c''(x)}{c'(x)} = -\frac{1}{x}$$

$$\frac{d}{dx}\ln c'(x) = -\frac{1}{x}$$

Integrate both sides with respect to x.

$$\ln c'(x) = -\ln x + C_1$$

Exponentiate both sides.

$$\begin{aligned} c'(x) &= e^{-\ln x + C_1} \\ &= e^{\ln x^{-1} + C_1} \\ &= e^{\ln x^{-1}} e^{C_1} \\ &= x^{-1} e^{C_1} \\ &= \frac{e^{C_1}}{x} \end{aligned}$$

Integrate both sides with respect to x once more.

$$c(x) = e^{C_1} \ln|x| + C_2$$

Using a new constant C_3 for e^{C_1}, the general solution is

$$\begin{aligned} y(x) &= c(x)x^2 \\ &= (C_3 \ln|x| + C_2)x^2 \\ &= C_3 x^2 \ln|x| + C_2 x^2. \end{aligned}$$

Because the initial condition is given at $x = -1$, replace $|x|$ with $-x$ for the moment.

$$y(x) = C_3 x^2 \ln(-x) + C_2 x^2$$

Differentiate it with respect to x.

$$y'(x) = 2C_3 x \ln(-x) + C_3 x^2 \cdot \frac{-1}{-x} + 2C_2 x$$

Now apply the initial conditions, $y(-1) = 2$ and $y'(-1) = 3$, to determine C_2 and C_3.

$$\begin{aligned} y(-1) &= C_2 = 2 \\ y'(-1) &= -C_3 - 2C_2 = 3 \end{aligned}$$

Solving this system of equations yields $C_2 = 2$ and $C_3 = -7$. Therefore,

$$y(x) = -7x^2 \ln|x| + 2x^2$$

Take the limit as $x \to 0$.

$$\lim_{x\to 0} y(x) = \lim_{x\to 0} -7x^2 \ln|x| + \underbrace{\lim_{x\to 0} 2x^2}_{=0} = -7 \lim_{x\to 0} \frac{\ln|x|}{\frac{1}{x^2}} \overset{\frac{\infty}{\infty}}{\underset{\text{H}}{=}} -7 \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{2}{x^3}} = \frac{7}{2} \lim_{x\to 0} x^2 = 0$$

Below is a plot of $y(x)$ versus x.