Problem 18

In each of Problems 17 through 34, find all singular points of the given equation and determine whether each one is regular or irregular.

$$x^2(1-x)^2y'' + 2xy' + 4y = 0$$

Solution

The coefficient of y'' has zeros at $x = 0$ and $x = 1$, which means $x = 0$ and $x = 1$ are singular points. To determine whether they are regular or irregular, divide both sides of the ODE by $x^2(1-x)^2$

$$y'' + \frac{2}{x(1-x)^2}y' + \frac{4}{x^2(1-x)^2}y = 0$$

and compute the following limits.

$$\lim_{x\to 0} x\frac{2}{x(1-x)^2} = \lim_{x\to 0} \frac{2}{(1-x)^2} = 2$$
$$\lim_{x\to 0} x^2\frac{4}{x^2(1-x)^2} = \lim_{x\to 0} \frac{4}{(1-x)^2} = 4$$

$$\lim_{x\to 1}(x-1)\frac{2}{x(1-x)^2} = \lim_{x\to 1} \frac{2}{x(1-x)} = \infty$$
$$\lim_{x\to 1}(x-1)^2\frac{4}{x^2(1-x)^2} = \lim_{x\to 1} \frac{4}{x^2} = 4$$

Because both limits as $x \to 0$ are finite, $x = 0$ is a regular singular point. However, because one of the limits as $x \to 1$ is infinite, $x = 1$ is an irregular singular point.