Problem 19

In each of Problems 17 through 34, find all singular points of the given equation and determine whether each one is regular or irregular.

$$x^2(1-x)y'' + (x-2)y' - 3xy = 0$$

Solution

The coefficient of y'' has zeros at $x = 0$ and $x = 1$, which means $x = 0$ and $x = 1$ are singular points. To determine whether they are regular or irregular, divide both sides of the ODE by $x^2(1-x)$

$$y'' + \frac{x-2}{x^2(1-x)}y' - \frac{3}{x(1-x)}y = 0$$

and compute the following limits.

$$\lim_{x\to 0} x\frac{x-2}{x^2(1-x)} = \lim_{x\to 0}\frac{x-2}{x(1-x)} = \infty$$
$$\lim_{x\to 0} x^2\frac{x-2}{x^2(1-x)} = \lim_{x\to 0}\frac{x-2}{1-x} = -2$$

$$\lim_{x\to 1}(x-1)\frac{x-2}{x^2(1-x)} = \lim_{x\to 1}\frac{2-x}{x^2} = 1$$
$$\lim_{x\to 1}(x-1)^2\frac{x-2}{x^2(1-x)} = \lim_{x\to 1}\frac{(1-x)(x-2)}{x^2} = 0$$

Because one of the limits as $x \to 0$ is infinite, $x = 0$ is an irregular singular point. However, because both limits as $x \to 1$ are finite, $x = 1$ is a regular singular point.