# Problem 24

In each of Problems 17 through 34, find all singular points of the given equation and determine whether each one is regular or irregular.

$$x(1-x^2)^3y'' + (1-x^2)^2y' + 2(1+x)y = 0$$

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### Solution

The coefficient of $y''$ has zeros at $x = 0$ and $x = 1$ and $x = -1$, which means $x = 0$ and $x = 1$ and $x = -1$ are singular points. To determine whether they are regular or irregular, divide both sides of the ODE by $x(1-x^2)^3$

$$y'' + \frac{(1-x^2)^2}{x(1-x^2)^3}y' + \frac{2(1+x)}{x(1-x^2)^3}y = 0$$

$$y'' + \frac{1}{x(1-x^2)}y' + \frac{2(1+x)}{x(1-x)^3(1+x)^3}y = 0$$

$$y'' + \frac{1}{x(1-x)(1+x)}y' + \frac{2}{x(1-x)^3(1+x)^2}y = 0$$

and compute the following limits.

$$\lim_{x\to 0} x\frac{1}{x(1-x)(1+x)} = \lim_{x\to 0} \frac{1}{(1-x)(1+x)} = 1$$

$$\lim_{x\to 0} x^2\frac{2}{x(1-x)^3(1+x)^2} = \lim_{x\to 0} \frac{2x}{(1-x)^3(1+x)^2} = 0$$

$$\lim_{x\to 1}(x-1)\frac{1}{x(1-x)(1+x)} = \lim_{x\to 1}(1-x)\frac{-1}{x(1-x)(1+x)} = \lim_{x\to 1}\frac{-1}{x(1+x)} = -\frac{1}{2}$$

$$\lim_{x\to 1}(x-1)^2\frac{2}{x(1-x)^3(1+x)^2} = \lim_{x\to 1}(1-x)^2\frac{2}{x(1-x)^3(1+x)^2} = \lim_{x\to 1}\frac{2}{x(1-x)(1+x)^2} = \infty$$

$$\lim_{x\to -1}(x+1)\frac{1}{x(1-x)(1+x)} = \lim_{x\to -1}(1+x)\frac{1}{x(1-x)(1+x)} = \lim_{x\to -1}\frac{1}{x(1-x)} = -\frac{1}{2}$$

$$\lim_{x\to -1}(x+1)^2\frac{2}{x(1-x)^3(1+x)^2} = \lim_{x\to -1}(1+x)^2\frac{2}{x(1-x)^3(1+x)^2} = \lim_{x\to -1}\frac{2}{x(1-x)^3} = -\frac{1}{4}$$

Because at least one of the limits as $x \to 1$ is infinite, $x = 1$ is an irregular singular point. However, because both limits as $x \to 0$ and $x \to -1$ are finite, $x = 0$ and $x = -1$ are regular singular points.