Problem 25

In each of Problems 17 through 34, find all singular points of the given equation and determine whether each one is regular or irregular.

$$(x+2)^2(x-1)y'' + 3(x-1)y' - 2(x+2)y = 0$$

Solution

The coefficient of y'' has zeros at $x = 1$ and $x = -2$, which means $x = 1$ and $x = -2$ are singular points. To determine whether they are regular or irregular, divide both sides of the ODE by $(x+2)^2(x-1)$

$$y'' + \frac{3}{(x+2)^2}y' - \frac{2}{(x+2)(x-1)}y = 0$$

and compute the following limits.

$$\lim_{x\to 1}(x-1)\frac{3}{(x+2)^2} = \lim_{x\to 1}\frac{3(x-1)}{(x+2)^2} = 0$$

$$\lim_{x\to 1}(x-1)^2\left[-\frac{2}{(x+2)(x-1)}\right] = \lim_{x\to 1}\frac{-2(x-1)}{(x+2)} = 0$$

$$\lim_{x\to -2}(x+2)\frac{3}{(x+2)^2} = \lim_{x\to -2}\frac{3}{x+2} = \infty$$

$$\lim_{x\to -2}(x+2)^2\left[-\frac{2}{(x+2)(x-1)}\right] = \lim_{x\to -2}\frac{-2(x+2)}{x-1} = 0$$

Because at least one of the limits as $x \to -2$ is infinite, $x = -2$ is an irregular singular point. However, because both limits as $x \to 1$ are finite, $x = 1$ is a regular singular point.