Problem 27

In each of Problems 17 through 34, find all singular points of the given equation and determine whether each one is regular or irregular.

$$(x^2 + x - 2)y'' + (x + 1)y' + 2y = 0$$

Solution

The coefficient of y'' has zeros at $x = 1$ and $x = -2$, which means $x = 1$ and $x = -2$ are singular points. To determine whether they are regular or irregular, divide both sides of the ODE by $x^2 + x - 2$

$$y'' + \frac{x+1}{x^2+x-2}y' + \frac{2}{x^2+x-2}y = 0$$

$$y'' + \frac{x+1}{(x+2)(x-1)}y' + \frac{2}{(x+2)(x-1)}y = 0$$

and compute the following limits.

$$\lim_{x\to 1}(x-1)\frac{x+1}{(x+2)(x-1)} = \lim_{x\to 1}\frac{x+1}{x+2} = \frac{2}{3}$$

$$\lim_{x\to 1}(x-1)^2\frac{2}{(x+2)(x-1)} = \lim_{x\to 1}\frac{2(x-1)}{x+2} = 0$$

$$\lim_{x\to -2}(x+2)\frac{x+1}{(x+2)(x-1)} = \lim_{x\to -2}\frac{x+1}{x-1} = \frac{1}{3}$$

$$\lim_{x\to -2}(x+2)^2\frac{2}{(x+2)(x-1)} = \lim_{x\to -2}\frac{2(x+2)}{x-1} = 0$$

Because both limits as $x \to 1$ and $x \to -2$ are finite, $x = 1$ and $x = -2$ are regular singular points.