Problem 1

In each of Problems 1 through 12, determine the general solution of the given differential equation that is valid in any interval not including the singular point.

\[ x^2y'' + 4xy' + 2y = 0 \]

Solution

This is a homogeneous equidimensional (Euler) ODE, so the solution is of the form \( y = x^r \).

\[ y = x^r \quad \Rightarrow \quad y' = rx^{r-1} \quad \Rightarrow \quad y'' = r(r-1)x^{r-2} \]

Substitute these expressions into the ODE.

\[ x^2r(r-1)x^{r-2} + 4rx^{r-1} + 2x^r = 0 \]

\[ r(r-1)x^r + 4rx^r + 2x^r = 0 \]

Divide both sides by \( x^r \).

\[ r(r-1) + 4r + 2 = 0 \]

Solve for \( r \).

\[ r^2 + 3r + 2 = 0 \]

\[ (r+2)(r+1) = 0 \]

\[ r = \{-2,-1\} \]

Two solutions to the ODE are \( y = x^{-2} \) and \( y = x^{-1} \). According to the principle of superposition, the general solution is a linear combination of these two. Therefore,

\[ y(x) = C_1x^{-2} + C_2x^{-1}. \]